REPRESENTATIONS OF FINITE DIMENSIONAL POINTED HOPF ALGEBRAS OVER $\mathbb{S}_3$

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ABSTRACT. The classification of finite-dimensional pointed Hopf algebras with group $\mathbb{S}_3$ was finished in [AHS]: there are exactly two of them, the bosonization of a Nichols algebra of dimension 12 and a non-trivial lifting. Here we determine all simple modules over any of these Hopf algebras. We also find the Gabriel quivers, the projective covers of the simple modules, and prove that they are not of finite representation type. To this end, we first investigate the modules over some complex pointed Hopf algebras defined in the papers [AG1, GG], whose restriction to the group of group-likes is a direct sum of 1-dimensional modules.

1. INTRODUCTION

In [AG1], a pointed Hopf algebra H_n was defined for each $n \geq 3$. It was shown there that H_3 and H_4 are non-trivial pointed Hopf algebras over $\mathbb{S}_3$ and $\mathbb{S}_4$, respectively. We showed in [GG] that this holds for every n, by different methods. We started by defining generic families of pointed Hopf algebras associated to certain data, which includes a finite non-abelian group G. Under certain conditions, these algebras are liftings of (possibly infinite dimensional) quadratic Nichols algebras over G. In particular, this was proven to hold for $G = \mathbb{S}_n$. Moreover, the classification of finite dimensional pointed Hopf algebras over $\mathbb{S}_4$ was finished. We review some of these facts in Section 2. We investigate, in Section 3, modules over these algebras whose G-isotypic components are 1-dimensional and classify indecomposable modules of this kind. We find conditions on a given G-character under which it can be extended to a representation of the algebra. We apply these results to the representation theory of two families of pointed Hopf algebras over $\mathbb{S}_n$. In Section 4 we comment on some known facts about simple modules over bosonizations. We also prove general facts about projective modules over the algebras defined in [AG1, GG], and recall a few facts about representation type of finite dimensional algebras. In Section 5 we use some of the previous results to classify simple modules over pointed Hopf algebras over $\mathbb{S}_3$. In addition, we find their projective covers and compute their fusion rules, which lead to show that the non-trivial lifting is not quasitriangular. We

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also write down the Gabriel quivers and show that these algebras are not of finite representation type.

2. Preliminaries

We work over an algebraically closed field $\Bbbk$ of characteristic zero. We fix $\mathrm{i} = \sqrt{-1}$. For $n \in \mathbb{N}$, let $[\frac{n}{2}]$ denote the biggest integer lesser or equal than $\frac{n}{2}$. If V is a vector space and $\{x_i\}_{i\in I}$ is a family of elements in V, we denote by $\Bbbk\{x_i\}_{i\in I}$ the vector subspace generated by it. Let G be a finite group, $\widehat{G}$ the set of its irreducible representations. Let $G_{\mathrm{ab}} = G/[G,G]$, $\widehat{G_{\mathrm{ab}}} = \mathrm{Hom}(G, \Bbbk^*) \subseteq \widehat{G}$. We denote by $\epsilon \in \widehat{G_{\mathrm{ab}}}$ the trivial representation. If $\chi \in \widehat{G}$, and W is a G-module, we denote by $W[\chi]$ the isotypic component of type χ, and by W_χ the corresponding simple G-module.

A *rack* is a pair $(X, \rhd)$, where X is a non-empty set and $\rhd : X \times X \to X$ is a function, such that $\phi_i = i \rhd (\cdot) : X \to X$ is a bijection for all $i \in X$ and $i \rhd (j \rhd k) = (i \rhd j) \rhd (i \rhd k)$, $\forall i, j, k \in X$. A rack $(X, \rhd)$ is said to be *indecomposable* if it cannot be decomposed as the disjoint union of two sub-racks. We shall always work with racks that are in fact quandles, that is that $i \rhd i = i \;\forall\, i \in X$. In practice, we are interested in the case in which the rack X is a conjugacy class in a group; hence this assumption always holds. We will denote by $\mathcal{O}_2^n$ the conjugacy class of transpositions in $\mathbb{S}_n$.

A 2-cocycle $q : X \times X \to \Bbbk^*$, $(i,j) \mapsto q_{ij}$ is a function such that $q_{i,j\rhd k}q_{j,k} = q_{i\rhd j, i\rhd k}q_{i,k}$, $\forall\, i,j,k \in X$. See [AG1] for a detailed exposition on this matter.

Let H be a Hopf algebra over $\Bbbk$, with antipode $\mathcal{S}$. Let ${}^H_H\mathcal{YD}$ be the category of (left-left) Yetter-Drinfeld modules over H. That is, M is an object of ${}^H_H\mathcal{YD}$ if and only if there exists an action $\cdot$ such that $(M, \cdot)$ is a (left) H-module and a coaction δ such that (M, δ) is a (left) H-comodule, subject to the following compatibility condition:

$$\delta(h \cdot m) = h_1 m_{-1} \mathcal{S}(h_3) \otimes h_2 \cdot m_0, \;\forall\, m \in M, h \in H,$$

where $\delta(m) = m_{-1} \otimes m_0$. If G is a finite group and $H = \Bbbk G$, we write ${}^G_G\mathcal{YD}$ instead of ${}^H_H\mathcal{YD}$.

Recall from [AG2, Def. 3.2] that a *principal YD-realization* of (X, q) over a finite group G is a collection $(\cdot, g, (\chi_i)_{i\in X})$ where

- $\cdot$ is an action of G on X;
- $g : X \to G$ is a function such that $g_{h\cdot i} = h g_i h^{-1}$ and $g_i \cdot j = i \rhd j$;
- the family $(\chi_i)_{i\in X}$, with $\chi_i : G \to \Bbbk^*$, is a 1-cocycle, *i. e.* $\chi_i(ht) = \chi_i(t)\chi_{t\cdot i}(h)$, for all $i \in X$, $h, t \in G$, satisfying $\chi_i(g_j) = q_{ji}$.

In words, a principal YD-realization over G is a way to realize the braided vector space $(\Bbbk X, c^q)$ as a YD-module over G. See [AG2] for details.

2.1. Quadratic lifting data.

Let X be a rack, q a 2-cocycle. Let $\mathcal{R}$ be the set of equivalence classes in $X \times X$ for the relation generated by $(i,j) \sim (i \rhd j, i)$. Let $C \in \mathcal{R}$, $(i,j) \in C$.

Take $i_1 = j$, $i_2 = i$, and recursively, $i_{h+2} = i_{h+1} \triangleright i_h$. Set $n(C) = \#C$ and

$$\mathcal{R}' = \Big\{ C \in \mathcal{R} \mid \prod_{h=1}^{n(C)} q_{i_{h+1},i_h} = (-1)^{n(C)} \Big\}.$$

Let F be the free associative algebra in the variables $\{T_l\}_{l\in X}$. If $C \in \mathcal{R}'$, consider the quadratic polynomial

$$\phi_C = \sum_{h=1}^{n(C)} \eta_h(C)\, T_{i_{h+1}} T_{i_h} \in F, \tag{1}$$

where $\eta_1(C) = 1$ and $\eta_h(C) = (-1)^{h+1} q_{i_2 i_1} q_{i_3 i_2} \dots q_{i_h i_{h-1}}$, $h \geq 2$.

A *quadratic lifting datum* $\mathcal{Q} = (X, q, G, (\cdot, g, (\chi_l)_{l\in X}), (\lambda_C)_{C\in\mathcal{R}'})$, or ql-datum, [GG, Def. 3.5], is a collection consisting of

- a rack X;
- a 2-cocycle q;
- a finite group G;
- a principal YD-realization $(\cdot, g, (\chi_l)_{l\in X})$ of (X, q) over G such that $g_i \neq g_j g_k$, $\forall\, i, j, k \in X$;
- a collection $(\lambda_C)_{C\in\mathcal{R}'} \in \Bbbk$ such that, if $C = \{(i_2, i_1), \dots, (i_n, i_{n-1})\}$, and $k \in X$,

$$\lambda_C = 0, \qquad \text{if } \ g_{i_2} g_{i_1} = 1, \tag{2}$$

$$\lambda_C = q_{k i_2} q_{k i_1} \lambda_{k \triangleright C}, \tag{3}$$

where $k \triangleright C = \{(k \triangleright i_2, k \triangleright i_1), \dots, (k \triangleright i_n, k \triangleright i_{n-1})\}$.

In [GG], we attached a pointed Hopf algebra $\mathcal{H}(\mathcal{Q})$ to each ql-datum $\mathcal{Q}$. It is generated by $\{a_l, H_t : l \in X,\ t \in G\}$ with relations:

$$H_e = 1, \quad H_t H_s = H_{ts}, \qquad t, s \in G; \tag{4}$$

$$H_t a_l = \chi_l(t) a_{t\cdot l} H_t, \qquad t \in G,\ l \in X; \tag{5}$$

$$\phi_C(\{a_l\}_{l\in X}) = \lambda_C (1 - H_{g_i g_j}), \qquad C \in \mathcal{R}',\ (i, j) \in C. \tag{6}$$

Here ϕ_C is as in (1) above. We denote by a_C the left-hand side of (6). $\mathcal{H}(\mathcal{Q})$ is a pointed Hopf algebra, setting $\Delta(H_t) = H_t \otimes H_t$, $\Delta(a_i) = g_i \otimes a_i + a_i \otimes 1$, $t \in G$, $i \in X$. See [GG] for further details on this construction and for unexplained terminology.

Notice that by definition of the Hopf algebras $\mathcal{H}(\mathcal{Q})$, the group of group-likes $G(\mathcal{H}(\mathcal{Q}))$ is a quotient of the group G. Thus, any $\mathcal{H}(\mathcal{Q})$-module M is a G-module, using the corresponding projection. We denote this module by $M_{|G}$. For simplicity, we denote $M[\rho] = M_{|G}[\rho]$, $\rho \in \widehat{G}$.

3. Modules that are sums of 1-dimensional representations

In this Section, we study $\mathcal{H}(\mathcal{Q})$-modules whose underlying G-module is a direct sum of representations in $\widehat{G_{\mathrm{ab}}}$.

We begin by fixing the following notation. Given a pair (X,q), let

$$\zeta_h(C) = \begin{cases} (-1)^{\frac{h}{2}-1} \left(\prod_{l=1}^{\frac{h}{2}-1} q_{i_{h-2l+1}, i_{h-2l}} \right) & \text{if } 2|h, \\ (-1)^{\frac{h-1}{2}} \left(\prod_{l=1}^{\frac{h-1}{2}} q_{i_{h-2l+1}, i_{h-2l}} \right) & \text{if } 2|h+1. \end{cases} \tag{7}$$

Note that $\zeta_1(C) = \zeta_2(C) = 1$, $\zeta_{h+1}(C)\zeta_h(C) = \eta_h(C)$, see (1).

3.1. Modules whose underlying G-module is isotypical.

We first study extensions of multiplicative characters from G to $\mathcal{H}(\mathcal{Q})$.

Proposition 3.1. *Let $\rho \in \widehat{G_{ab}}$. There exists $\bar\rho \in \hom_{alg}(\mathcal{H}(\mathcal{Q}), \Bbbk)$ such that $\bar\rho_{|G} = \rho$ if and only if*

$$0 = \lambda_C(1 - \rho(g_i g_j)) \ \textit{if } (i,j) \in C \textit{ and } 2|n(C), \tag{8}$$

and there exists a family $\{\gamma_i\}_{i\in X}$ of scalars such that

$$\gamma_j = \chi_j(t)\gamma_{t\cdot j} \qquad \forall\, t \in G, j \in X, \tag{9}$$

$$\gamma_i\gamma_j = \lambda_C(1 - \rho(g_i g_j)) \qquad \textit{if } (i,j) \in C \textit{ and } 2|n(C)+1. \tag{10}$$

If (8) holds, then the set of all extensions $\bar\rho$ of ρ is in bijective correspondence with the set of families $\{\gamma_i\}_{i\in X}$ that satisfy (9) and (10). In particular, if

$$\lambda_C \neq 0 \Rightarrow \rho(g_i g_j) = 1, \quad C \in \mathcal{R}',\ (i,j) \in C. \tag{11}$$

then $\gamma_i = 0$, $\forall\, i \in X$ defines an $\mathcal{H}(\mathcal{Q})$-module. Moreover, this is the only possible extension if, in addition,

$$\chi_i(g_i) \neq 1, \quad \forall\, i \in X. \tag{12}$$

Remark 3.2. (a) Mainly, we will deal with Nichols algebras for which the following is satisfied:

$$\chi_i(g_i) = -1, \quad \forall\, i \in X. \tag{13}$$

In this case, obviously (12) holds and the class $C_i = \{(i,i)\}$ belongs to $\mathcal{R}'$.

(b) If X is indecomposable, using (9) and the fact that $\forall\, i \in X\ \exists\, t \in G$ such that $i = t \cdot j$, we may replace (10) by

$$\gamma_j^2 = \lambda_C(1 - \rho(g_j)^2)\chi_j(t) \qquad \text{if } (i,j) \in C \text{ and } 2|n(C)+1. \tag{10'}$$

Proof. Assume that such $\bar\rho$ exists and let $\gamma_i = \bar\rho(a_i)$. Then (9) follows from (5). In particular, for $p,q \in X$, we have $\bar\rho(a_{p\triangleright q}) = \chi_q(g_p)^{-1}\bar\rho(a_q)$. Then, for $C \in \mathcal{R}'$, $(i_2, i_1) = (i,j) \in C$, it follows that

$$\gamma_{i_h} = \bar\rho(a_{i_h}) = \begin{cases} (-1)^{\frac{h-1}{2}}\zeta_h(C)^{-1}\bar\rho(a_j) & \text{if } 2|h+1 \\ (-1)^{\frac{h}{2}-1}\zeta_h(C)^{-1}\bar\rho(a_i) & \text{if } 2|h, \end{cases} \tag{14}$$

cf. (7). Consequently,

$$\bar\rho(a_{i_{h+1}}a_{i_h}) = (-1)^{h+1}\eta_h(C)^{-1}\bar\rho(a_i)\bar\rho(a_j) \tag{15}$$

and thus (10) and (8) follow from (6). Conversely, if (8) holds and $\{\gamma_i\}_{i\in X}$ is a family that satisfies (9) and (10), then we define $\bar\rho : \mathcal{H}(\mathcal{Q}) \to \Bbbk$ as the unique algebra morphism such that $\bar\rho(H_t) = \rho(t)$ and $\bar\rho(a_i) = \gamma_i$. If (12) holds, it follows from (9) for $t = g_i$ that $\bar\rho(a_i) = 0\,\forall\, i \in X$ is a necessary condition. □

Definition 3.3. Let $\bar\rho$ be an extension of $\rho \in \widehat{G_{ab}}$ and $\gamma_i = \bar\rho(a_i)$, $\gamma = (\gamma_i)_{i\in X} \in \Bbbk^X$. Then we denote the corresponding $\mathcal{H}(\mathcal{Q})$-module by S_ρ^γ. If $\gamma = 0$, we set $S_\rho^\gamma = S_\rho$.

We now determine all $\mathcal{H}(\mathcal{Q})$-modules whose underlying G-module is isotypical of type $\rho \in \widehat{G_{ab}}$, provided that X is indecomposable and (12) holds.

Proposition 3.4. *Assume X is indecomposable. Let M be an $\mathcal{H}(\mathcal{Q})$-module such that $M = M[\rho]$ for a unique $\rho \in \widehat{G_{ab}}$, $\dim M = n$. Then M is simple if and only if $n = 1$. If, in addition, (12) holds, $M \cong S_\rho^{\oplus n}$.*

Proof. Let $\bar\rho : \mathcal{H}(\mathcal{Q}) \to \operatorname{End} M$ be the corresponding representation and $\Gamma_j \in \Bbbk^{n\times n}$ be the matrix associated to $\bar\rho(a_j)$ in some (fixed) basis. As in the proof of Prop. 3.1, $\{\Gamma_i\}_{i\in X}$ satisfies (9). Thus, if we fix $j \in X$, then for each $i \in X$ there exists $t \in G$ such that $\Gamma_i = \chi_j(t)^{-1}\Gamma_j$. Thus, there exists a basis $\{z_1, \dots, z_n\}$ in which all of these matrices are upper triangular and so $\Bbbk\{z_1\}$ generates a submodule $M' \subseteq M$. If (12) holds, then it follows that $\Gamma_i = 0$, $\forall\, i \in X$ and thus $M \cong \bigoplus_{j=1}^n S_\rho$. □

3.2. Modules whose underlying G-module is a sum of two isotypical components.

Let $\rho, \mu \in \widehat{G_{ab}}$ fulfilling (8), $\gamma, \delta \in \Bbbk^X$ satisfying (9) and (10) for ρ and μ, respectively. We begin this Subsection by describing indecomposable modules that are extensions of S_ρ^γ by S_μ^δ. For simplicity of the statement of (17) in the following Lemma, we introduce the following notation. Let $C \in \mathcal{R}'$, $j \in C$ and let

$$\alpha_j(C) = \sum_{r=0}^{[\frac{n(C)}{2}]-1} \chi_j(g_j)^r, \quad \beta_j(C) = \sum_{r=0}^{[\frac{n(C)+1}{2}]-1} \chi_j(g_j)^r.$$

Note that if $2|n(C)$, then $\alpha_j = \beta_j$; otherwise, $\beta_j = \alpha_j + \chi_j(g_j)^{[\frac{n(C)+1}{2}]-1}$.

Lemma 3.5. *Let V be the space of solutions $\{f_i\}_{i\in X} \in \Bbbk^X$ of the following system*

$$f_i\mu(t) = \chi_i(t) f_{t\cdot i}\rho(t), \qquad i \in X,\ t \in G \text{ and} \tag{16}$$

$$(\alpha_j(C)\delta_j - \beta_j(C)\gamma_j)f_i = -\chi_i(g_i)(\alpha_i(C)\delta_i - \beta_i(C)\gamma_i)f_j, \tag{17}$$

$C \in \mathcal{R}'$, $(i,j) \in C$. *Then* $\operatorname{Ext}^1_{\mathcal{H}(\mathcal{Q})}(S^\gamma_\rho, S^\delta_\mu) \cong V$ *and the set of isomorphism classes of indecomposable* $\mathcal{H}(\mathcal{Q})$*-modules such that*

$$0 \longrightarrow S^\delta_\mu \longrightarrow M \longrightarrow S^\gamma_\rho \longrightarrow 0 \text{ is exact} \tag{18}$$

is in bijective correspondence with $\mathbb{P}_k(V)$*.*

Proof. Let $M = \Bbbk\{z, w\}$ be as in (18), with $z \in M[\rho]$, $w \in M[\mu]$. Then there exists $\{f_i\}_{i\in X}$ such that

$$a_i z = \gamma_i z + f_i w. \tag{19}$$

Then (16) follows from (5) and this implies

$$f_{i_h} = \begin{cases} (-\chi_j(g_j))^{\frac{h}{2}-1}\zeta_h(C)^{-1} f_i & \text{if } 2|h, \\ (-\chi_i(g_i))^{\frac{h-1}{2}}\zeta_h(C)^{-1} f_j & \text{if } 2|h+1, \end{cases}$$

since, for $\tau = \rho$ or $\tau = \mu$,

$$\tau(g_{i_{2l+1}}) = \tau(g_{i_{2l}} g_{i_{2l-1}} g_{i_{2l}}^{-1}) = \tau(g_{i_{2l-1}}) = \cdots = \tau(g_{i_1}) = \tau(g_j),$$
$$\tau(g_{i_{2l+2}}) = \tau(g_{i_{2l+1}} g_{i_{2l}} g_{i_{2l+1}}^{-1}) = \tau(g_{i_{2l}}) = \cdots = \tau(g_{i_2}) = \tau(g_i),$$

and $\frac{\mu(g_k)}{\rho(g_k)} = \chi_k(g_k)$. Therefore, if $(i,j) \in C$ and $n = n(C)$, (6) holds if and only if

$$\sum_{h=1}^{n} \eta_h(C)\left(f_{i_h}\delta_{i_{h+1}} + f_{i_{h+1}}\gamma_{i_h}\right) = 0, \forall\, C \in \mathcal{R}',$$

that is, using (14), (6) holds if and only if (17) follows.

Conversely, if $\{f_i\}_{i\in X}$ fulfills (16) and (17), then (19) together with $a_i w = \delta_i w$ define an $\mathcal{H}(\mathcal{Q})$-module which is an extension of S^γ_ρ by S^δ_μ.

M is indecomposable if and only if $f_i \neq 0$ for some $i \in X$. Assume M is indecomposable and let $M' = \Bbbk\{z', w'\}$ be another indecomposable $\mathcal{H}(\mathcal{Q})$-module fitting in (18), with $z' \in M'[\rho]$, $w' \in M'[\mu]$. Let $\{g_i\}_{i\in X} \in V$ be the corresponding solution of (16) and (17). Assume $\phi : M \to M'$ is an isomorphism of $\mathcal{H}(\mathcal{Q})$-modules. In particular, ϕ is a G-isomorphism and thus there exist $\sigma, \tau \in \Bbbk^*$ such that $\phi(w) = \sigma w'$, $\phi(z) = \tau z'$. But then it is readily seen that σ, τ must satisfy $g_i = \sigma\tau^{-1} f_i$, $i \in X$. That is, $[f_i]_{i\in X} = [g_i]_{i\in X}$ in $\mathbb{P}_\Bbbk(V)$. The converse is clear. □

Remark 3.6. If X is indecomposable, then, up to isomorphism, there is at most one indecomposable $\mathcal{H}(\mathcal{Q})$-module M as in the Lemma. In fact, if there is one, let $\{f_i\}_{i\in X} \in \Bbbk^X$ be the corresponding solution of (16) and (17). Then, if we fix $j \in X$ and let $t_i \in G$ be such that $i = t_i \cdot j$, $i \in X$, then

$$(f_i)_{i\in X} = f_j \left(\chi_j(t_i)\frac{\mu(t_i)}{\rho(t_i)}\right)_{i\in X} \in \Bbbk^X, \tag{20}$$

and thus M is uniquely determined. In this case, the existence of a solution is equivalent to (16) and

$$(\alpha_j\delta_j - \beta_j\gamma_j)\left(\frac{\mu(t_i)}{\rho(t_i)} + \chi_j(g_j)\right) f_j = 0; \tag{17'}$$

if $(i,j) \in C$, $C \in \mathcal{R}'$, $i = t_i \cdot j$.

Definition 3.7. Assume X is indecomposable and $\operatorname{Ext}^1_{\mathcal{H}(\mathcal{Q})}(S^\gamma_\rho, S^\delta_\mu) \neq 0$. We denote the corresponding unique indecomposable $\mathcal{H}(\mathcal{Q})$-module by $M^{\gamma,\delta}_{\rho,\mu}$. If $\gamma = \delta = 0$, then (17') is a tautology. We set $M_{\rho,\mu} := M^{0,0}_{\rho,\mu}$.

Assume that X is indecomposable and that $G = \langle \{g_i\}_{i\in X}\rangle$. Let j be a fixed element in X. Define $\ell : G \to \mathbb{Z}$, resp. $\psi : G \to \Bbbk^*$, as

$$\ell(t) = \min\{n \,:\, t = g_{i_1} \dots g_{i_n},\ i_1, \dots, i_n \in X\},$$

resp. $\psi(t) = \chi_j(g_j)^{\ell(t)}$, $t \in G$. Notice that $\tau(g_i) = \tau(g_j)$, $\forall\, i \in X$, hence $\tau(t) = \tau(g_j)^{\ell(t)}$, for any $\tau \in \widehat{G_{ab}}$, $t \in G$.

Lemma 3.8. *Keep the above hypotheses. If $\operatorname{Ext}^1_{\mathcal{H}(\mathcal{Q})}(S^\gamma_\rho, S^\delta_\mu) \neq 0$, then*

$$\mu(s) = \psi(s)\rho(s), \qquad \forall s \in G. \tag{21}$$

Therefore ρ determines μ (and vice versa), and ψ is a group homomorphism.

Conversely, if (21) holds, we may replace (16) and (17) by

$$f_i\chi_j(g_j)^{\ell(t)} = \chi_i(t) f_{t\cdot i}, \qquad i \in X,\ t \in G \text{ and} \tag{16'}$$

$$0 = f_j(\alpha_j\delta_j - \beta_j\gamma_j)\left(\chi_j(g_j)^{\ell(t_i)-1} + 1\right), \tag{17''}$$

if $(i,j) \in C$, $C \in \mathcal{R}'$, $i = t_i \cdot j$.

Proof. Setting $i = j$ and $t = g_j$ in (16), and taking the $\ell(s)$-th power, we get (21). The rest is straightforward. □

We will show next that there are no simple modules M of dimension 2 such that $M_{|G}$ is sum of two (necessarily different) components of dimension 1, provided that the following holds:

$$\exists\, C \in \mathcal{R}' \quad \text{with} \quad n(C) > 1. \tag{22}$$

Notice that if (22) does not hold and $\operatorname{gr}\mathcal{H}(\mathcal{Q}) = \mathfrak{B}(X,q)\sharp\Bbbk G$, then it follows that $\dim\mathcal{H}(\mathcal{Q}) = \infty$, provided that $|X| > 1$, since $\{(a_ia_j)^n\}_{n\in\mathbb{N}}$ is a linearly independent set in $\mathcal{H}(\mathcal{Q})$.

Lemma 3.9. *Assume X is indecomposable, and that* (13) *and* (22) *hold. Let $\rho, \mu \in \widehat{G_{ab}}$, and let M be an $\mathcal{H}(\mathcal{Q})$-module such that $M = M[\rho] \oplus M[\mu]$, $\dim M[\rho] = \dim M[\mu] = 1$. Then M is not simple.*

Proof. Assume that there exists M simple as in the hypothesis. We first claim that $\rho \neq \mu$ and that, if $z \in M[\rho]$, then $a_iz \in M[\mu]$. In fact, let $a_iz = u + w$ with $u \in M[\rho]$, $w \in M[\mu]$, then

$$H_ta_iz = \rho(t)u + \mu(t)w, \quad \chi_i(t)a_{t\cdot i}H_tz = \chi_i(t)\rho(t)a_{t\cdot i}z$$

and taking $t = g_i$, we get

$$\rho(g_i)u + \mu(g_i)w = \chi_i(g_i)\rho(g_i)(u+w) \overset{(13)}{=} -\rho(g_i)u - \rho(g_i)w.$$

Thus $u = 0$; hence $w \neq 0$ because M is simple. Also,

$$\rho(g_i) = -\mu(g_i), \quad i \in X. \tag{23}$$

By a symmetric argument, $a_i(M[\mu]) = M[\rho]$.

Now, fix $0 \neq z \in M[\rho]$, $0 \neq w \in M[\mu]$; let f_i, $i \in X$, such that $a_i z = f_i w$. Then $(f_i)_{i \in X}$ satisfies (16), by (5). As X is indecomposable and M is simple, we have $f_i \neq 0$, $\forall\, i \in X$. We necessarily have

$$a_i w = p_i z, \quad \text{for} \quad p_i = f_i^{-1}\lambda_i(1-\rho(g_i)^2). \tag{24}$$

Note that $p_i \neq 0$ or otherwise $a_i w = 0$, $\forall\, i \in X$. As stated for $\{f_i\}$, the family $\{p_i\}$ also satisfies (16), with the roles of ρ and μ interchanged.

Assume that there is $C \in \mathcal{R}'$, with $n(C) > 1$. We now show that this contradicts the existence of M. Let $(i_2, i_1) = (i, j) \in C$, then

$$a_C z = \sum_{h=1}^{n(C)} \eta_h f_{i_h} a_{i_{h+1}} w = \sum_{h=1}^{n(C)} \eta_h f_{i_h} \frac{\lambda_{i_{h+1}}}{f_{i_{h+1}}}(1-\rho(g_{i_{h+1}})^2) z.$$

Let $t \in G$ such that $i = t \cdot j$ and recall that $i_h = i_{h-1} \triangleright i_{h-2}$. Since $g_{s\cdot k} = g_s g_k g_s^{-1}$, then

$$\rho(g_{i_{h+1}})^2 = \rho(g_j)^2, \quad \forall\, h.$$

Now, by (3), $\lambda_{i_h} = \lambda_{i_{h-1} \triangleright i_{h-2}} = \chi_{i_{h-2}}(g_{i_{h-1}})^{-2}\lambda_{i_{h-2}}$, then

$$\lambda_{i_h} = \begin{cases} \zeta_h(C)^{-2}\chi_j(t)^{-2}\lambda_j & \text{if } 2|h, \\ \zeta_h(C)^{-2}\lambda_j & \text{if } 2|h+1. \end{cases}$$

Additionally, by (16), we have

$$f_{i_h} = \begin{cases} \zeta_h(C)^{-1}\chi_j(t)^{-1}\dfrac{\mu(t)}{\rho(t)} f_j & \text{if } 2|h, \\ \zeta_h(C)^{-1} f_j & \text{if } 2|h+1, \end{cases} \tag{25}$$

for every $h = 1, \dots, n(C)$. Therefore, we have that:

$$\eta_h(C)\lambda_{i_{h+1}}\frac{f_{i_h}}{f_{i_{h+1}}} = \begin{cases} \dfrac{\mu(t)}{\rho(t)}\chi_j(t)^{-1}\lambda_j & \text{if } 2|h, \\[2ex] \dfrac{\rho(t)}{\mu(t)}\chi_j(t)^{-1}\lambda_j & \text{if } 2|h+1. \end{cases} \tag{26}$$

Analogously, if we analyze the element $a_C w$, we get

$$\eta_h(C)\lambda_{i_{h+1}}\frac{p_{i_h}}{p_{i_{h+1}}} = \begin{cases} \dfrac{\rho(t)}{\mu(t)}\chi_j(t)^{-1}\lambda_j & \text{if } 2|h, \\[2ex] \dfrac{\mu(t)}{\rho(t)}\chi_j(t)^{-1}\lambda_j & \text{if } 2|h+1. \end{cases} \tag{27}$$

However, notice that, if $h > 1$,

$$\begin{aligned}\eta_h(C)\lambda_{i_{h+1}}\frac{p_{i_h}}{p_{i_{h+1}}} &= \eta_h(C)\lambda_{i_{h+1}}\frac{\lambda_{i_h}(1-\rho(g_{i_h})^2)f_{i_{h+1}}}{\lambda_{i_{h+1}}(1-\rho(g_{i_{h+1}})^2)f_{i_h}}\\ &= -\eta_{h-1}(C)\chi_{i_{h-1}}(g_{i_h})\lambda_{i_h}\frac{f_{i_{h+1}}}{f_{i_h}} \overset{(16)}{=} -\eta_{h-1}(C)\lambda_{i_h}\frac{f_{i_{h-1}}}{f_{i_h}}\frac{\mu(t)}{\rho(t)}\\ &\overset{(26)}{=} \begin{cases} -\dfrac{\mu(t)^2}{\rho(t)^2}\chi_j(t)^{-1}\lambda_j & \text{if } 2|h-1,\\ \\ -\chi_j(t)^{-1}\lambda_j & \text{if } 2|h.\end{cases}\end{aligned}$$

And from this equality together with (27), we get

$$\rho(t) = -\mu(t), \quad \text{if} \quad (i,j)\in C, \quad t\cdot j = i. \tag{28}$$

But, as $i \triangleright i = i$, we have that $\mu(g_it) = -\rho(g_it)$ and also

$$\mu(g_it) = \mu(g_i)\mu(t) \overset{(23)}{=} -\rho(g_i)\mu(t) = \rho(g_i)\rho(t) = \rho(g_it),$$

which is a contradiction.

□

Assume X is indecomposable. Next, we describe indecomposable modules which are sums of two different isotypical components, provided that (13) and (22) hold.

Theorem 3.10. *Let $\rho \neq \mu \in \widehat{G_{ab}}$. Assume X is indecomposable and both* (13) *and* (22) *hold. Let $M = M[\rho]\oplus M[\mu]$ be an $\mathcal{H}(\mathcal{Q})$-module, with* $\dim M[\rho]$, $\dim M[\mu] > 0$. *Then M is not simple.*

Moreover, M is a direct sum of modules of the form S_ρ^γ, S_μ^δ, $M_{\rho,\mu}^{\gamma',\delta'}$ and $M_{\mu,\rho,}^{\delta'',\gamma''}$ for various γ,δ, γ',δ', γ'',δ''.

Proof. Take $0\neq z\in M[\rho]$. As in the first part of the proof of Lemma 3.9, it follows from (13) that $\rho\neq\mu$ and that, if $0\neq z\in M[\rho]$, then $a_iz\in M[\mu]$. Now, $a_iw = a_i^2z = \lambda_i(1-\rho(g_i)^2)z$, and thus the space $\Bbbk\{z,w\}$ is a_i-stable. As X is indecomposable, it follows that this is a submodule. Let $K = \ker a_i$. Here we see a_i as an operator in $\operatorname{End} M$. This subspace is G-stable: if $u\in K$, $u = z+w$, with $z\in M[\rho]$, $w\in M[\mu]$, then $0 = a_iu = a_iz + a_iw \Rightarrow z,w\in K$, since $a_iw\in M[\rho]$, $a_iz\in M[\mu]$. Thus $\rho(t)z = H_tz$ and $\mu(t)w = H_tw\in K$, $\forall\, t\in G$. Therefore $G\cdot u\subset K$. The same holds for $I = \operatorname{im} a_i$. Let T be a G-submodule such that $M = K\oplus T$ (recall $\Bbbk G$ is semisimple). Let

$$K = \ker a_i = K[\rho]\oplus K[\mu], \quad T = T[\rho]\oplus T[\mu], \quad I = \operatorname{im} a_i = I[\rho]\oplus I[\mu].$$

Notice that $K\neq 0$. In fact, if $K = 0$, then the space $\Bbbk\{z,w\}$ would be a simple 2-dimensional $\mathcal{H}(\mathcal{Q})$-module, contradicting Lemma 3.9. Thus $K\neq 0$. Then $\gamma_i = 0$, $\forall\, i\in X$ and $a_i^2\cdot M = 0$. Notice that in this case $I[\psi]\subseteq K[\psi]$,

for $\psi = \rho$ or μ, and thus we have $K[\psi] = I[\psi] \oplus J[\psi]$. As G-modules, we have

$$M_{|G} \cong \bigoplus_{\psi=\rho,\mu} M[\psi] = \bigoplus_{\psi=\rho,\mu} I[\psi] \oplus J[\psi] \oplus T[\psi],$$

and this induces the following decomposition of $\mathcal{H}(\mathcal{Q})$-modules:

$$M \cong J[\rho] \oplus J[\mu] \oplus (I[\rho] + T[\mu]) \oplus (I[\mu] + T[\rho]).$$

Let $\psi = \rho$ or μ. If $J[\psi] \neq 0$, then (8) holds for ψ, and $J[\psi]$ is a sum of 1-dimensional $\mathcal{H}(\mathcal{Q})$-modules, by Prop. 3.4. Let $\{w_1, \dots, w_k\}$ be a basis of $T[\mu]$. Then $\{a_i w_1, \dots, a_i w_k\}$ is a basis of $I[\rho]$. In fact, if $z \in I[\rho]$, $z = a_i w$, $w \in T[\mu]$, there are $\sigma_1, \dots, \sigma_k \in \Bbbk$ such that $w = \sum_{j=1}^k \sigma_j w_j$ and then $z = \sum_{j=1}^k \sigma_j a_i w_j$. If, on the other hand, $\{\sigma_j\}_{j=1}^k \in \Bbbk$ satisfy $0 = \sum_{j=1}^k \sigma_j a_i w_j$ then $\sum_{j=1}^k \sigma_j w_j \in K[\mu]$, and as $K \cap T = 0$, $\sigma_j = 0\, \forall\, j = 1, \dots, k$. Thus $I[\rho] + T[\mu] = \bigoplus_{j=1}^k \langle w_j \rangle$ as $\mathcal{H}(\mathcal{Q})$-modules. By Lemma 3.5, for each $j = 1, \dots, k$ there exists $\delta_j, \gamma_j \in \Bbbk^{*X}$ such that $\langle w_j \rangle \cong M_{\mu,\rho}^{\delta_j,\gamma_j}$. A similar statement follows for $I[\mu] + T[\rho]$. Therefore, there are $m_\rho, m_\mu, m_{\rho,\mu}, m_{\mu,\rho} \in \mathbb{N}_0$, $\{\xi_j\}_{j=1}^{m_\rho}, \{\pi_j\}_{j=1}^{m_\mu}, \{\delta_j\}_{j=1}^{m_{\rho,\mu}}, \{\gamma_j\}_{j=1}^{m_{\rho,\mu}}, \{\sigma_j\}_{j=1}^{m_{\mu,\rho}}, \{\tau_j\}_{j=1}^{m_{\mu,\rho}} \in \Bbbk^X$ such that

$$M \cong \bigoplus_{j=1}^{m_\rho} S_\rho^{\xi_j} \oplus \bigoplus_{j=1}^{m_\mu} S_\rho^{\pi_j} \oplus \bigoplus_{j=1}^{m_{\rho,\mu}} M_{\mu,\rho}^{\delta_j,\gamma_j} \oplus \bigoplus_{j=1}^{m_{\mu,\rho}} M_{\mu,\rho}^{\sigma_j,\tau_j},$$

where m_ρ (resp. m_μ) is non-zero only if (8) holds for ρ (resp. μ), ξ_j, π_j and satisfy (9) and (10) for ρ, μ respectively. On the other hand, $m_{\rho,\mu} \neq 0$ only if (16) holds for ρ, μ and δ_j, γ_j satisfy (17). Similarly for $m_{\mu,\rho}$, σ_j, τ_j. $\square$

3.3. The case $G = \mathbb{S}_n$, $n \geq 3$.

Let $\Lambda, \Gamma, \lambda \in \Bbbk$, $t = (\Lambda, \Gamma)$, $\iota : \mathcal{O}_2^n \hookrightarrow \mathbb{S}_n$ the inclusion, $\cdot : \mathbb{S}_n \times X \to X$ the action given by conjugation, -1 the constant cocycle $q \equiv -1$ and χ the cocycle given by, if $\tau, \sigma \in \mathcal{O}_2^n$, $\tau = (ij)$ and $i < j$:

$$\chi(\sigma, \tau) = \begin{cases} 1, & \text{if } \sigma(i) < \sigma(j) \\ -1, & \text{if } \sigma(i) > \sigma(j), \end{cases} \qquad \text{see [MS, Ex. 5.3].}$$

Then the ql-data:

- $\mathcal{Q}_n^{-1}[t] = (\mathbb{S}_n, \mathcal{O}_2^n, -1, \cdot, \iota, \{0, \Lambda, \Gamma\})$, $n \geq 4$;
- $\mathcal{Q}_n^{\chi}[\lambda] = (\mathbb{S}_n, \mathcal{O}_2^n, \chi, \cdot, \iota, \{0, 0, \lambda\})$, $n \geq 4$;
- $\mathcal{Q}_3^{-1}[\lambda] = (\mathbb{S}_3, \mathcal{O}_2^3, -1, \cdot, \iota, \{0, \lambda\})$;

define pointed Hopf algebras over $\mathbb{S}_n$, for n as appropriate, [AG2, GG].

Remark 3.11. Notice that the racks $\mathcal{O}_2^n$, $n \geq 3$ are indecomposable and that (13) is satisfied for both cocycles. In this case, $\widehat{G_{ab}} = \{\epsilon, \mathrm{sgn}\}$, where ϵ, resp. sgn, stands for the trivial, resp. sign, representation. In any case, (11) holds. Bear also in mind that $\mathbb{S}_n = \langle \mathcal{O}_2^n \rangle$. In this case, the function

$\ell : G \to \mathbb{Z}$ is well-known and $\psi : G \to \{\pm 1\} \subset \Bbbk^*$ coincides with the sign function, by (13). Moreover, (22) holds in all of these ql-data.

Proposition 3.12. *Let $A = \mathcal{H}(\mathcal{Q}_n^{-1}[t])$ or $\mathcal{H}(\mathcal{Q}_3^{-1}[\lambda])$. Let M be an A-module such that $M_{|\mathbb{S}_n} = M[\epsilon] \oplus M[\mathrm{sgn}]$, $\dim M[\epsilon] = p$, $\dim M[\mathrm{sgn}] = q$. Then*

- (i) *M is simple if and only if $M = S_\epsilon$ or $M = S_{\mathrm{sgn}}$.*
- (ii) *M is indecomposable if and only if M is simple or $p = q = 1$. In this last case, there are two non-isomorphic indecomposable modules, namely $M_{\epsilon,\mathrm{sgn}}$ and $M_{\mathrm{sgn},\epsilon}$.*

Proof. It follows by Props. 3.1 and 3.4, and by Lemma 3.9 that S_ϵ and S_{sgn} are the unique two simple modules. The second item follows by Thm. 3.10 and Lemma 3.8. $\square$

Proposition 3.13. *Let $n \geq 4$. Let M be a $\mathcal{H}(\mathcal{Q}_n^{\chi}[\lambda])$-module such that $M_{|\mathbb{S}_n} = M[\epsilon] \oplus M[\mathrm{sgn}]$, with $\dim M[\epsilon] = p$, $\dim M[\epsilon] = q$, $p, q \geq 0$. Then M is indecomposable if and only if it is simple if and only if $M = S_\epsilon$ or $M = S_{\mathrm{sgn}}$.*

Proof. The determination of the simple modules follows from Props. 3.1 and 3.4 and Lemma 3.9. By Lemma 3.8 there are no extensions between 1-dimensional modules. Hence, the Prop. follows from Thm. 3.10. $\square$

4. General facts

Let H be a Hopf algebra, $V \in {}^H_H\mathcal{YD}$. The Nichols algebra $\mathfrak{B}(V) = \oplus_{n\geq 0}\mathfrak{B}^n(V)$ is a graded braided Hopf algebra in ${}^H_H\mathcal{YD}$ generated by V, in such a way that $V = \mathfrak{B}^1(V) = \mathcal{P}(\mathfrak{B}(V))$, that is, it is generated in degree one by its primitive elements which in turn coincide with the module V. This algebra is uniquely determined, up to isomorphism. See [AS] for details.

Let G be a finite group. Let X be a rack, q a 2-cocycle and assume that there exists a YD-realization of (X, q) over G. We denote by $\mathfrak{B}(X, q)$ the corresponding Nichols algebra.

4.1. Simple modules over bosonizations.

Consider the bosonization $\mathcal{A} = \mathfrak{B}(X, q)\sharp\Bbbk G$. As an algebra, $\mathcal{A}$ is generated by $\mathfrak{B}(X, q)$ and $\Bbbk G$; the product is defined by $(a\sharp t)(b\sharp s) = a(t \cdot b)\sharp ts$, here $\cdot$ stands for the action in ${}^G_G\mathfrak{YD}$. See [AS, 2.5] for details. In what follows, we shall assume that $\mathfrak{B}(X, q)$, and thus $\mathcal{A}$, is finite dimensional. The following proposition is well-known. We state it and prove it here for the sake of completeness.

Proposition 4.1. *The simple modules for $\mathcal{A}$ are in bijective correspondence with the simple modules over G: Given $\rho \in \widehat{G}$, S_ρ is the $\mathcal{A}$-module such that*

$$S_\rho \cong W_\rho \text{ as } G\text{-modules,} \quad \text{and} \quad a_i S_\rho = 0, \quad \forall\, i \in X.$$

This correspondence preserves tensor products and duals.

Proof. With the action stated above, it is clear that for each $\rho \in \widehat{G}$, S_ρ is an $\mathcal{A}$-module. If $\mathfrak{B}(X,q)^+$ denotes the maximal graded ideal of $\mathfrak{B}(X,q)$, then the Jacobson radical $J = J(\mathcal{A})$ is given by $J = \mathfrak{B}(X,q)^+\sharp\Bbbk G$. In fact J is a maximal nilpotent ideal (since $\mathcal{A}$ is graded and finite dimensional) and $\mathcal{A}/J \cong \Bbbk G$ is semisimple. This also shows that the list $\{S_\rho : \rho \in \widehat{G}\}$ is an exhaustive list of $\mathfrak{B}(X,q)$-modules, which are obviously pairwise non-isomorphic. The last assertion follows since $a_i\,(S_\rho \otimes S_\mu) = 0$ and $\mathcal{S}(a_i) = -H_{g_i}^{-1}a_i$. □

4.2. **Projective covers of modules over quadratic liftings.**

Let B be a ring, M a left B-module. A *projective cover* of M is a pair $(P(M), f)$ with $P = P(M)$ a projective B-module and $f : P \to M$ an *essential map*, that is f is surjective and for every $N \subset M$ proper submodule, $f(N) \neq M$. We will not explicit the map f when it is obvious. Projective covers are unique up to isomorphism and always exist for finite-dimensional $\Bbbk$-algebras, see [CR, Sect. 6]. Moreover,

$$ {}_B B \cong \bigoplus_{S\in\widehat{B}} P(S)^{\dim S}. \tag{29}$$

Fix G a finite group and H a pointed Hopf algebra over G. Let $\{e_i\}_{i=1}^N$ be a complete set of orthogonal primitive idempotents for G and set $I_j = He_j$, for $1 \le j \le N$.

Lemma 4.2. *$I_j = \operatorname{Ind}_{\Bbbk G}^H \Bbbk Ge_j$. In particular, if $\Bbbk Ge_j \cong \Bbbk Ge_h$ as G-modules, then $I_j \cong I_h$ as H-modules.*

Moreover, $H \cong \bigoplus_{\rho\in\widehat{G}} I_\rho^{\dim\rho}$ as H-modules, where $I_\rho = \operatorname{Ind}_{\Bbbk G}^H W_\rho$, and thus I_ρ is a projective H-module.

Proof. Let $\psi : \operatorname{Ind}_{\Bbbk G}^H \Bbbk Ge_j \to H$ be the composition of the multiplication $m : H \otimes_{\Bbbk G} \Bbbk G \to H$ with the inclusion $H \otimes_{\Bbbk G} \Bbbk Ge_j \to H \otimes_{\Bbbk G} \Bbbk G$. It follows that $\operatorname{im}\psi = I_j$. Then $I_j = \operatorname{Ind}_{\Bbbk G}^H \Bbbk Ge_j$ and I_j does not depend on the idempotent e_j but on the simple module $W_\rho = \Bbbk Ge_j$. Therefore, as $\Bbbk G = \oplus_{i=1}^N \Bbbk Ge_i$, we have that $H \cong \bigoplus_{\rho\in\widehat{G}} I_\rho^{\dim\rho}$. □

Let $\{H_n\}_{n\in\mathbb{N}}$ be the coradical filtration of H,

$$\operatorname{gr}^n H = H_n/H_{n-1}, \quad \operatorname{gr} H = \oplus_{n\ge 0} \operatorname{gr}^n H.$$

We know that there exists $R \in {}^G_G\mathcal{YD}$ such that $\operatorname{gr} H \cong R\sharp\Bbbk G$, see [AS, 2.7]. Let $\pi_n : H_n \to \operatorname{gr}^n H$ be the canonical projection. As every H_n is $\operatorname{ad}(G)$-stable, it follows that π_n is a morphism of G-modules. Therefore there exists a section $\operatorname{gr}^n H \to H_n$ and $H_n \cong \operatorname{gr}^n H \oplus H_{n-1}$ as G-modules. By an inductive argument we have that $H_n \cong \operatorname{gr}^n H \oplus \operatorname{gr}^{n-1} H \oplus \cdots \oplus \operatorname{gr}^0 H$. And thus it follows that $H \cong \operatorname{gr} H$ as G-modules. Moreover, it follows that, if we consider the adjoint action on $\Bbbk G$, $\operatorname{gr} H \cong R \otimes \Bbbk G$ as G-modules, via the diagonal action. Thus, $H \cong R \otimes \Bbbk G$ as G-modules.

Proposition 4.3. *Let $\operatorname{gr} H = R\sharp\Bbbk G$.*

(i) *$I_\epsilon \cong R$ as G-modules.*
(ii) *Assume there exists a simple H-module M such that $M_{|\Bbbk G}$ is a simple G-module W_ρ. Then $P(M)$ is a direct summand of I_ρ. In particular, if I_ρ is indecomposable, then $I_\rho \cong P(M)$.*
(iii) *If $H = R\sharp\Bbbk G$, I_ρ is the projective cover of S_ρ, see Prop. 4.1.*

Proof. Let W_ϵ be the trivial G-module. Since $I_\epsilon = \mathrm{Ind}_{\Bbbk G}^{H} W_\epsilon$ and $H \cong R \otimes \Bbbk G$, we have

$$(I_\epsilon)_{|G} \cong ((R \otimes \Bbbk G) \otimes_{\Bbbk G} W_\epsilon)_{|G} \cong R_{|G}.$$

Thus the first item follows. Let now M be an H-module such that $M_{|\Bbbk G} = W_\rho$. If $(P(M), f)$ is the projective cover of M, we have the commutative diagram:

$$\begin{tikzcd} & I_\rho \arrow[dl, dashed, "\tau"'] \arrow[d, "\pi"] \\ P(M) \arrow[r, two heads, "f"] & M \end{tikzcd}$$

where $\pi : I_\rho \to M$ is the factorization of the action $\cdot : H \otimes M \to M$ through $H \otimes M \twoheadrightarrow I_\rho = H \otimes_{\Bbbk G} W_\rho$. As $f(\tau(I_\rho)) = \pi(I_\rho) = M$ and f is essential, we have an epimorphism $I_\rho \twoheadrightarrow P(M)$ and $P(M)$ is a direct summand of I_ρ. Thus $I_\rho \cong P(M)$, if I_ρ is assumed to be indecomposable.

Finally, assume $H = R\sharp\Bbbk G$. If $P(S_\rho)$ is the projective cover of S_ρ, we must have $\dim P(S_\rho) \leq \dim I_\rho = \dim R \dim W_\rho$. But we see that this is in fact an equality from the formulas:

$$\dim H = \dim R \sum_{\rho\in\widehat{G}} \dim W_\rho^2 = \sum_{\rho\in\widehat{G}} (\dim R \dim W_\rho) \dim W_\rho$$
$$\dim H = \sum_{\rho\in\widehat{G}} \dim P(S_\rho) \dim S_\rho = \sum_{\rho\in\widehat{G}} \dim P(S_\rho) \dim W_\rho.$$

□

4.3. **Representation type.**

We comment on some general facts about the representation type of a finite dimensional algebra, that will be employed in 5.2.2 and 5.3.6. Let B be a finite dimensional $\Bbbk$-algebra, $\widehat{B} = \{S_1, \dots, S_n\}$ a complete list of non-isomorphic simple B-modules. The *Ext-Quiver* (also *Gabriel quiver*) of B is the quiver $ExtQ(B)$ with vertices $\{1, \dots, n\}$ and $\dim \mathrm{Ext}^1_B(S_i, S_j)$ arrows from the vertex i to the vertex j. Then B is Morita equivalent to the basic algebra $\Bbbk ExtQ(B)/I(B)$, where $\Bbbk ExtQ(B)$ is the path algebra of the quiver $ExtQ(B)$ and $I(B)$ is an ideal contained in the bi-ideal of paths of length greater than one. Recall that for any two B modules M_1, M_2 there is an isomorphism of abelian groups

$$\mathrm{Ext}^1_B(M_1, M_2) = \{\text{equivalence classes of extensions of } M_1 \text{ by } M_2\},$$

where the element 0 is given by the trivial extension $M_1 \oplus M_2$.

Given a quiver Q with vertices $V = \{1, \dots, n\}$, its *separation diagram* is the unoriented graph with vertices $\{1', \dots, n', 1'', \dots, n''\}$ and with an edge i'—j'' for each arrow $i \to j$ in Q. If B is algebra, we speak of the separation diagram of B referring to the separation diagram of its Ext-Quiver.

Theorem 4.4. [ARS, Th. 2.6] *Let B be an Artin algebra with radical square zero. Then B is of finite (tame) representation type if and only if its separated diagram is a disjoint union of finite (affine) Dynkin diagrams.* $\square$

Lemma 4.5. *Let J be the radical of B. Then $ExtQ(B) = ExtQ(B/J^2)$.*

Proof. First, it is immediate that $\widehat{B} = \widehat{B/J^2}$. Let $S, T \in \widehat{B}$. As any B/J^2-module is a B-module, we have $\mathrm{Ext}^1_{B/J^2}(S,T) \subseteq \mathrm{Ext}^1_B(S,T)$. Now, let

$$0 \to T \hookrightarrow V \twoheadrightarrow S \to 0 \in B - \mathrm{mod}, \quad x \in V, a_1, a_2, \in J.$$

If $x \in T \subset V$, then $a_1 x = 0 \Rightarrow a_2 a_1 x = 0$. If $x \notin T$, then $0 \neq \bar{x} \in V/T \cong S$ and thus $a_1 \bar{x} = 0$, that is $a_1 x \in T$, and therefore $a_2 a_1 x = 0$. Thus, the above exact sequence in $B - \mathrm{mod}$ gives rise to an exact sequence in $B/J^2 - \mathrm{mod}$, proving the lemma. $\square$

5. Representation theory of pointed Hopf algebras over $\mathbb{S}_3$

In this Section we investigate the representations of the finite dimensional pointed Hopf algebras over $\mathbb{S}_3$. We will denote by $\mathcal{A}_\lambda$, $\lambda \in \Bbbk$, the algebra $\mathcal{H}((\mathcal{Q}_3^{-1}[\lambda]))$. This algebra was introduced in [AG1]. Explicitly, it is generated by elements H_t, a_i, $t, i \in \mathcal{O}_2^3$; with relations

$$\begin{aligned} H_t H_s H_t = H_s H_t H_s,\ H_t^2 = 1, &\qquad s \neq t \in \mathcal{O}_2^3; \\ H_t a_i = -a_{t\sigma i} H_t, &\qquad t, i \in \mathcal{O}_2^3; \\ a_{12}^2 = 0, & \\ a_{12}a_{23} + a_{23}a_{13} + a_{13}a_{12} = \lambda(1 - H_{12}H_{23}). & \end{aligned}$$

$\mathcal{A}_\lambda$ is a Hopf algebra of dimension 72. If H is a finite-dimensional pointed Hopf algebra with $G(H) \cong \mathbb{S}_3$, then either $H \cong \Bbbk\mathbb{S}_3$, $H \cong \mathcal{A}_0$ or $H \cong \mathcal{A}_1$ [AHS, Theorem 4.5], together with [MS, AG1, AZ].

We will determine all simple modules over $\mathcal{A}_0$ and $\mathcal{A}_1$, along with their projective covers and fusion rules. We will also show that these algebras are not of finite representation type and classify indecomposable modules satisfying certain restrictions.

Remark 5.1. Notice that to describe an $\mathcal{A}_\lambda$-module supported on a given G-module, it is enough to describe the action of a_{12}, since $a_{13}, a_{23} \in \mathrm{ad}(G)(a_{12})$.

5.1. Simple $\Bbbk\mathbb{S}_3$-modules.

We will need some facts about the representation theory of $\mathbb{S}_3$, which we state next. Besides the modules W_ϵ and W_{sgn} associated to the characters ϵ and sgn, respectively, there is one more simple $\Bbbk\mathbb{S}_3$-module, namely the standard representation W_{st}. This module

has dimension 2. We fix $\{v, w\}$ as its canonical basis. In this basis the representation is given by the following matrices:

$$[H_{12}] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad [H_{23}] = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \quad [H_{13}] = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Given a $\Bbbk\mathbb{S}_3$-module W, we denote by $W[\mathrm{st}]$ the isotypical component corresponding to this representation.

5.2. **Representation theory of $\mathcal{A}_0$.**

Proposition 5.2. *There are exactly three simple $\mathcal{A}_0$-modules, namely the extensions S_ϵ, S_{sgn} and S_{st} of the simple $\Bbbk\mathbb{S}_3$-modules.*

Proof. Follows from Prop. 4.1. □

5.2.1. *Some indecomposable $\mathcal{A}_0$-modules.*
Fix $\langle x\rangle_{\mathbb{S}_3} = W_\epsilon$, $\langle y\rangle_{\mathbb{S}_3} = W_{\mathrm{sgn}}$, $\langle v, w\rangle_{\mathbb{S}_3} = W_{\mathrm{st}}$.

Lemma 5.3. *There are exactly four non-isomorphic non-simple indecomposable $\mathcal{A}_0$-modules of dimension 3:*

(i) $M_{\mathrm{st},\epsilon} = \Bbbk\{x, v, w\}$, *with* $a_{12}\cdot v = x$, $a_{12}\cdot x = 0$;

(ii) $M_{\mathrm{st},\mathrm{sgn}} = \Bbbk\{y, v, w\}$, *with* $a_{12}\cdot v = y$, $a_{12}\cdot y = 0$;

(iii) $M_{\epsilon,\mathrm{st}} = \Bbbk\{x, v, w\}$, *with* $a_{12}\cdot x = v - w$, $a_{12}\cdot v = 0$;

(iv) $M_{\mathrm{sgn},\mathrm{st}} = \Bbbk\{y, v, w\}$, *with* $a_{12}\cdot y = v + w$, $a_{12}\cdot v = 0$.

In particular, $\dim \mathrm{Ext}^1_{\mathcal{A}_0}(S_{\mathrm{st}}, S_\sigma) = \dim \mathrm{Ext}^1_{\mathcal{A}_0}(S_\sigma, S_{\mathrm{st}}) = 1$*,* $\sigma \in \{\epsilon, \mathrm{sgn}\}$.

Proof. By Prop. 3.12, we know that such an $\mathcal{A}_0$-module M must contain a copy of W_{st}. Thus $M_{|\mathbb{S}_3} \cong W_\epsilon \oplus W_{\mathrm{st}}$ or $M_{|\mathbb{S}_3} \cong W_{\mathrm{sgn}} \oplus W_{\mathrm{st}}$. The lemma now follows by straightforward computations. □

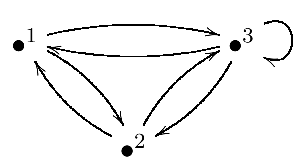
Proposition 5.4. *The non-isomorphic indecomposable modules which are extensions of S_{st} by itself are indexed by $\mathbb{P}^1_{\Bbbk}$. In particular, it follows that* $\dim \mathrm{Ext}^1_{\mathcal{A}_0}(S_{\mathrm{st}}, S_{\mathrm{st}}) = 1$.

Proof. If $\{v_1, v_2, w_1, w_2\}$ is basis of such a module, with $\{v_2, w_2\}_{|\mathbb{S}_3} = W_{\mathrm{st}}$, $\{v_1, w_1\} \cong M_{st}$, then a necessary condition is that $a_{12}v_2 = av_1 + bw_1$, $a \neq 0$ or $b \neq 0$. It is easy to see that this formula defines in fact an indecomposable $\mathcal{A}_0$ module $M_{(a,b)}$ for each (a, b) and that two of these modules, $M_{(a,b)}$ and $M_{(a',b')}$, are isomorphic if and only if $\exists\, \gamma \neq 0$ such that $(a, b) = \gamma(a', b')$. □

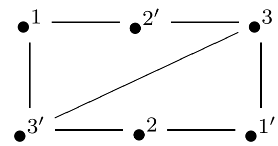
5.2.2. *Representation type of $\mathcal{A}_0$.*

Proposition 5.5. $\mathcal{A}_0$ *is of wild representation type.*

Proof. From Lemmas 3.8 and 5.3 together with Prop. 5.4, we see that the Ext-Quiver of $\mathcal{A}_0$ is



where we have ordered the simple modules as $\{S_\epsilon, S_{\rm sgn}, S_{\rm st}\} = \{1,2,3\}$. Thus, the separation diagram of $\mathcal{A}_0$ is



which implies that $\mathcal{A}_0$ is wild. □

5.3. **Representation theory of $\mathcal{A}_1$.**

We investigate now the simple modules of $\mathcal{A}_1$, their fusion rules and projective covers, and also the representation type of this algebra.

5.3.1. *Modules that are sums of 2-dimensional representations.* We first focus our attention on those $\mathcal{A}_1$-modules supported on sums of standard representations of $\Bbbk\mathbb{S}_3$.

Lemma 5.6. *Let $M_{\rm st} = \Bbbk\{v, w\}$. Then, the following formulas define four non-isomorphic $\mathcal{A}_1$-modules supported on $M_{\rm st}$:*

(i) $\quad a_{12}v = \mathrm{i}(v-w), \qquad a_{12}w = \mathrm{i}(v-w);$

(ii) $\quad a_{12}v = -\mathrm{i}(v-w), \qquad a_{12}w = -\mathrm{i}(v-w);$

(iii) $\quad a_{12}v = \dfrac{\mathrm{i}}{3}(v+w), \qquad a_{12}w = -\dfrac{\mathrm{i}}{3}(v+w);$

(iv) $\quad a_{12}v = -\dfrac{\mathrm{i}}{3}(v+w), \qquad a_{12}w = \dfrac{\mathrm{i}}{3}(v+w).$

They are simple modules, and we denote them by $S_{\rm st}(\mathrm{i}), S_{\rm st}(-\mathrm{i}), S_{\rm st}(\frac{\mathrm{i}}{3})$, $S_{\rm st}(-\frac{\mathrm{i}}{3})$, respectively.

Proof. Straightforward. □

Proposition 5.7. *Let $p \in \mathbb{N}$ and let M be an $\mathcal{A}_1$-module such that $M = M[\mathrm{st}]$, $\dim M = 2p$. Then M is completely reducible.*

M is simple if only if $p = 1$. In this case, it is isomorphic to one of the modules $S_{\rm st}(\mathrm{i}), S_{\rm st}(-\mathrm{i}), S_{\rm st}(\frac{\mathrm{i}}{3}), S_{\rm st}(-\frac{\mathrm{i}}{3})$.

Proof. Let $\{v_i, w_i\}_{i=1}^p$ be copies of the canonical basis of $W_{\rm st}$ such that $\{v_i, w_i\}_{i=1}^p$ is a linear basis of M. Let $v = (v_1, \dots, v_p)$, $w = (w_1, \dots, w_p)$. Now, there must exist matrices $\alpha, \beta \in \Bbbk^{p\times p}$ such that $a_{12}\cdot v = \alpha v + \beta w$ and thus $a_{12}\cdot w = -\beta v - \alpha w$, by acting with H_{12}. By acting with the rest of the elements H_t we get:

$$a_{13}\cdot v = -(\alpha+\beta)v + 2(\alpha+\beta)w, \qquad a_{13}\cdot w = -\beta v + (\alpha+\beta)w,$$
$$a_{23}\cdot v = -(\alpha+\beta)v + \beta w \qquad a_{23}\cdot w = -2(\alpha+\beta)v + (\alpha+\beta)w.$$

Now, $0 = a_{12}^2 v = \alpha a_{12} \cdot v + \beta a_{12} \cdot w = (\alpha^2 - \beta^2)v + (\alpha\beta - \beta\alpha)w$, and this implies that $\alpha^2 = \beta^2$, $\alpha\beta = \beta\alpha$. Hence,

$$(a_{12}a_{13} + a_{13}a_{23} + a_{23}a_{12}) \cdot v = (-5\alpha^2 - 4\alpha\beta)(v + w),$$
$$\text{while} \quad (1 - H_{12}H_{13}) \cdot v = v + w,$$

and thus $-5\alpha^2 - 4\alpha\beta = \mathrm{id}$.

Now, we have that, in particular, $-5\alpha - 4\beta = \alpha^{-1}$ and therefore $\beta = -\frac{5}{4}\alpha - \frac{1}{4}\alpha^{-1}$. Thus,

$$\alpha^2 = \beta^2 = \frac{1}{16}(5\alpha + \alpha^{-1})^2 = \frac{1}{16}(25\alpha^2 + \alpha^{-2} + 10\,\mathrm{id}),$$

from where it follows $(\alpha^2)^{-1} = -9\alpha^2 - 10\,\mathrm{id}$ and $\mathrm{id} = -9\alpha^4 - 10\alpha^2$, which is equivalent to

$$(\alpha^2 + \frac{5}{9}\,\mathrm{id})^2 = \frac{16}{81}\,\mathrm{id}\,. \tag{30}$$

This gives, in particular, that if $\theta \in \Bbbk$ is an eigenvalue of α, then $\theta \in L(\alpha) := \{\pm\mathrm{i}, \pm\frac{\mathrm{i}}{3}\}$. Now, let $\alpha \in \Bbbk^{p\times p}$ be a matrix satisfying equation (30). A simple analysis of the possible Jordan forms $J(\alpha)$ of α gives $J(\alpha) = \mathrm{diag}(\theta_1, \dots, \theta_p)$, for some $\theta_i \in L(\alpha)$, $i = 1, \dots, p$. If $p > 1$, we get that there is a basis of M in which α (and consequently β) is a diagonal matrix, and so M is completely reducible.

On the other hand, if $p = 1$, $\alpha \in L(\alpha)$ and $\beta = \pm\alpha$ give the module structures defined in Lemma 5.6. □

5.3.2. *Classification of simple modules over $\mathcal{A}_1$.* Now, we present the classification of all simple $\mathcal{A}_1$-modules.

Theorem 5.8. *Let M be a simple $\mathcal{A}_1$-module. Then M is isomorphic to one and only one of the following:*

- S_ϵ;
- S_{sgn};
- $S_{\mathrm{st}}(\mathrm{i})$, $S_{\mathrm{st}}(-\mathrm{i})$, $S_{\mathrm{st}}(\frac{\mathrm{i}}{3})$ *or* $S_{\mathrm{st}}(-\frac{\mathrm{i}}{3})$.

Proof. We know that the listed modules are all simple. In view of Props. 3.12 and 5.7, we are left to deal with the case in which $M_{|\mathbb{S}_3} = M[\epsilon] \oplus M[\mathrm{sgn}] \oplus M[\mathrm{st}]$, with $\dim M[\epsilon] = n$, $\dim M[\mathrm{sgn}] = m$, $\dim M[\mathrm{st}] = p$, $n + m, p > 0$. Let $\{x_1, \dots, x_n, y_1, \dots, y_m, v_1, \dots, v_p, w_1, \dots, w_p\}$ be a basis of M such that $\Bbbk\{x_i\} \cong W_\epsilon$, $i = 1, \dots, n$, $\Bbbk\{y_j\} \cong W_{\mathrm{sgn}}$, $j = 1, \dots, m$, $\Bbbk\{v_k, w_k\} \cong W_{\mathrm{st}}$, $k = 1, \dots, p$. Using the action of H_{12}, we find that there are matrices $\alpha \in \Bbbk^{n\times m}$, $\beta \in \Bbbk^{n\times p}$, $\gamma \in \Bbbk^{m\times n}$, $\eta \in \Bbbk^{m\times p}$, $a \in \Bbbk^{p\times n}$, $b \in \Bbbk^{p\times m}$ and $c, d \in \Bbbk^{p\times p}$, such that, if $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_m)$, $v = (v_1, \dots, v_p)$, $w = (w_1, \dots, w_p)$, the action of a_{12} is determined by the following equations:

$$a_{12} \cdot x = \alpha y + \beta(v - w), \qquad a_{12} \cdot y = \gamma x + \eta(v + w)$$
$$a_{12} \cdot v = ax + by + cv + dw, \qquad a_{12} \cdot w = -ax + by - dv - cw.$$

We deduce as in Prop. 5.7 the action of every a_σ:

$$\begin{aligned}
a_{13}\cdot x &= \alpha y - \beta v, & a_{13}\cdot v &= -2ax-(c+d)v+2(c+d)w,\\
a_{13}\cdot y &= \gamma x + \eta(v-2w) & a_{13}\cdot w &= -ax-by-dv+(c+d)w,\\
a_{23}\cdot x &= \alpha y + \beta w, & a_{23}\cdot v &= ax-by-(c+d)v+dw,\\
a_{23}\cdot y &= \gamma x + \eta(w-2v), & a_{23}\cdot w &= 2ax-2(c+d)v+(c+d)w.
\end{aligned}$$

Recall that it is enough to find a subspace stable under the action of a_{12} and the elements H_t, by Rem. 5.1. Now,

$$\begin{aligned}
0 = a_{12}^2 x &= (\alpha\gamma + 2\beta a)x + (\alpha\eta + \beta(c+d))(v+w);\\
0 = a_{12}^2 y &= (\gamma\alpha + 2\eta b)y + (\gamma\beta + \eta(c-d))(v-w);\\
0 = a_{12}^2 v &= (b\gamma + (c-d)a)x + (a\alpha + (c+d)b)y\\
&\quad + (a\beta + b\eta + c^2 - d^2)v + (-a\beta + b\eta + cd - dc)w;
\end{aligned}$$

$$\begin{aligned}
0 &= (a_{12}a_{13} + a_{13}a_{23} + a_{23}a_{12})\cdot x = (3\alpha\gamma - 3\beta a)x - 3\beta by;\\
0 &= (a_{12}a_{13} + a_{13}a_{23} + a_{23}a_{12})\cdot y = 9\eta ax + 3(\gamma\alpha - \eta b)y;\\
v + w &= (a_{12}a_{13} + a_{13}a_{23} + a_{23}a_{12})\cdot v\\
&= (-3a\beta - 3b\eta - c^2 - 4d^2 - 2dc - 2cd)v\\
&\quad + (3a\beta + 3b\eta - 4c^2 - d^2 - 2dc - 2cd)w.
\end{aligned}$$

Then we have the following equalities:

$$\left\{\begin{aligned}
&0 = \gamma\alpha = \alpha\gamma = \beta a = \beta b = \eta a = \eta b,\\
&\beta(c+d) + \alpha\eta = 0 = \eta(c-d) + \gamma\beta,\\
&b\gamma + (c-d)a = 0 = a\alpha + (c+d)b,\\
&d^2 - c^2 = a\beta + b\eta, \quad cd - dc = a\beta - b\eta\\
&3a\beta + 3b\eta = -c^2 - 4d^2 - 2dc - 2cd - \mathrm{id}\\
&3a\beta + 3b\eta = 4c^2 + d^2 + 2dc + 2cd + \mathrm{id}\,.
\end{aligned}\right. \tag{31}$$

From the last two equations:

$$c^2 - d^2 = 2(a\beta + b\eta), \quad 5(c^2 + d^2) + 4(dc + cd) = -2\,\mathrm{id},$$

and thus $a\beta + b\eta = 0$, $c^2 = d^2$. Notice that the matrix of a_{12} in the chosen basis is:

$$[a_{12}] = \begin{pmatrix} 0 & {}^t\gamma & {}^ta & -{}^ta\\ {}^t\alpha & 0 & {}^tb & {}^tb\\ {}^t\beta & {}^t\eta & {}^tc & -{}^td\\ -{}^t\beta & {}^t\eta & {}^td & -{}^tc \end{pmatrix}.$$

Now we make the following

Claim. *If α or γ have a null row, then M is not simple.*

In fact, assume $(\alpha_{11},\dots,\alpha_{1n})=0$. We have $a_{12}\cdot x_1=\sum_j\beta_{1j}(v_j-w_j)$, if this is zero, then $\langle x_1\rangle\cong S_\epsilon\subset M$ and M is not simple. If not, let

$$\bar v_1=\sum_j\beta_{1j}v_j,\quad \bar w_1=\sum_j\beta_{1j}w_j.$$

Thus, $a_{12}\cdot x_1=\bar v_1-\bar w_1$ and as $0=a_{12}^2x_1$ we have that $a_{12}\bar v_1=a_{12}\bar w_1$. But, moreover, we also have that

$$a_{12}\bar v_1=\sum_i(\beta a)_{1i}x_i+\sum_k(\beta(c+d))_{1k}(v_k+w_k)=0,$$

since $\beta a=0$ and $(\beta(c+d))_{1k}=-(\alpha\eta)_{1k}=-\sum_l\alpha_{1l}\eta_{lk}=0$. Then $\bar v_1=0$, $S_\epsilon\subset M$ and M is not simple.

The claim when a row of γ is null follows analogously, or just tensoring with the representation S_{sgn}, since it interchanges the roles of α and γ.

Then we see that, for M to be simple, we necessarily must have ${}^t\alpha$, ${}^t\gamma$ injective. But $0={}^t(\alpha\gamma)={}^t\gamma\,{}^t\alpha\Rightarrow\alpha=0$. Thus M cannot be simple if $n,m>0$. Therefore, we are left with the (equivalent) cases

$$M_{|\mathbb{S}_3}=M[\epsilon]\oplus M[\mathrm{st}],\qquad \text{with }\dim M[\epsilon]=n,\quad \dim M[\mathrm{st}]=p,\quad n,p>0;$$
$$M_{|\mathbb{S}_3}=M[\mathrm{sgn}]\oplus M[\mathrm{st}],\quad \text{with }\dim M[\mathrm{sgn}]=m,\quad \dim M[\mathrm{st}]=p,\ m,p>0.$$

Assume we are in the first case. Thus, the equations above become:

$$\left\{\begin{array}{l} a\beta=\beta a=0,\quad \beta(c+d)=0,\quad (c-d)a=0,\\ d^2=c^2,\quad cd=dc,\quad c(-5c-4d)=\mathrm{id}\,. \end{array}\right. \tag{32}$$

Now, in particular, if ${}^t\beta$ is injective, we have ${}^ta=0$ and thus $\mathcal{A}_1\cdot M[\mathrm{st}]\subsetneq M[\mathrm{st}]$. But if ${}^t\beta$ is not injective, we may find a non-trivial linear combination x of the elements $\{x_i\}_{i=1}^n$ making $S_\epsilon=\langle x\rangle$ into an $\mathcal{A}_1$-submodule of M. $\square$

5.3.3. *Some indecomposable $\mathcal{A}_1$-modules.*

We start by studying the 3-dimensional indecomposable modules. As said in Lemma 5.3, it follows that for such a module M, it holds either that $M_{|\mathbb{S}_3}\cong W_\epsilon\oplus W_{\mathrm{st}}$ or $M_{|\mathbb{S}_3}\cong W_{\mathrm{sgn}}\oplus W_{\mathrm{st}}$. Take x,y,v,w such that $\langle x\rangle_{|\mathbb{S}_3}=W_\epsilon$, $\langle y\rangle_{|\mathbb{S}_3}=W_{\mathrm{sgn}}$, $\langle v,w\rangle_{|\mathbb{S}_3}=W_{\mathrm{st}}$.

Lemma 5.9. *There are exactly eight non-isomorphic non-simple indecomposable $\mathcal{A}_1$-modules of dimension 3:*

(i) $M_{\mathrm{st},\epsilon}[\pm\frac{\mathrm{i}}{3}]=\Bbbk\{x,v,w\}$, $\quad a_{12}\cdot v=\pm\frac{\mathrm{i}}{3}(v+w)+x$, $\quad a_{12}\cdot x=0$;

(ii) $M_{\mathrm{st},\mathrm{sgn}}[\pm\mathrm{i}]=\Bbbk\{y,v,w\}$, $\quad a_{12}\cdot v=\pm\mathrm{i}(v-w)+y$, $\quad a_{12}\cdot y=0$;

(iii) $M_{\epsilon,\mathrm{st}}[\pm\mathrm{i}]=\Bbbk\{x,v,w\}$, $\quad a_{12}\cdot v=\pm\mathrm{i}(v-w)$, $\quad a_{12}\cdot x=v-w$;

(iv) $M_{\mathrm{sgn},\mathrm{st}}[\pm\frac{\mathrm{i}}{3}]=\Bbbk\{y,v,w\}$, $\quad a_{12}\cdot v=\pm\frac{\mathrm{i}}{3}(v+w)$, $\quad a_{12}\cdot y=v+w$.

Proof. It is straightforward to check that the listed objects are in fact $\mathcal{A}_1$-modules and that they are not isomorphic to each other. Now, assume $M_{|\mathbb{S}_3}=W_\epsilon\oplus W_{\mathrm{st}}$, the other case being analogous. If M is not simple, then

there is $N \subset M$ and necessarily $N_{|\mathbb{S}_3} = W_{\rm st}$ or $N_{|\mathbb{S}_3} = W_\epsilon$. Then, the lemma follows specializing the equations in (31) to this case. $\square$

Proposition 5.10. *Let M be an indecomposable non-simple $\mathcal{A}_1$-module such that $M_{|\mathbb{S}_3} = M[\epsilon] \oplus M[\mathrm{st}]$, with $\dim M[\epsilon] = p$, $\dim M[\mathrm{st}] = q$ or $M_{|\mathbb{S}_3} = M[\mathrm{sgn}] \oplus M[\mathrm{st}]$, with $\dim M[\mathrm{sgn}] = p$, $\dim M[\mathrm{st}] = q$ for $p, q > 0$. Then $p = q = 1$ and M is isomorphic to one and only one of the modules defined in Lemma 5.9.*

Proof. We work with the case $M_{|\mathbb{S}_3} = M[\epsilon] \oplus M[\mathrm{st}]$, with $\dim M[\epsilon] = p$, $\dim M[\mathrm{st}] = q$, $p, q \geq 1$, the other resulting from this one by tensoring with $S_{\rm sgn}$. Let $M[\epsilon] = \Bbbk\{x_i\}_{i=1}^p$, $M[st] = \Bbbk\{v_i, w_i\}_{i=1}^q$ and a, β, c, d be as in the proof of Th. 5.8. Recall that they satisfy the system of equations (32). The last three conditions from that system imply, as in the proof of Prop. 5.7, that c, d may be chosen as

$$c = \begin{pmatrix} \delta & 0 \\ 0 & \delta' \end{pmatrix}, \quad d = \begin{pmatrix} -\delta & 0 \\ 0 & \delta' \end{pmatrix},$$

for $\delta \in \Bbbk^{q_1 \times q_1}$, $\delta' \in \Bbbk^{q_2 \times q_2}$ diagonal matrices with eigenvalues in $\{\pm \mathrm{i}\}$ and $\{\pm \frac{\mathrm{i}}{3}\}$, respectively, $q_1 + q_2 = q$. Consequently,

$$\beta = \begin{pmatrix} \beta_1 & 0 \\ \beta_2 & 0 \end{pmatrix}, \quad a = \begin{pmatrix} 0 & 0 \\ a_1 & a_2 \end{pmatrix}, \text{ with } a_1\beta_1 + a_2\beta_2 = 0,$$

$$a_{12} = \begin{pmatrix} 0 & 0 & 0 & {}^t a_1 & 0 & -{}^t a_1 \\ 0 & 0 & 0 & {}^t a_2 & 0 & -{}^t a_2 \\ {}^t\beta_1 & {}^t\beta_2 & \delta & 0 & \delta & 0 \\ 0 & 0 & 0 & \delta' & 0 & -\delta' \\ -{}^t\beta_1 & -{}^t\beta_2 & -\delta & 0 & -\delta & 0 \\ 0 & 0 & 0 & \delta' & 0 & -\delta' \end{pmatrix}.$$

Assume $q_2 > 0$. In this case, $\tilde{a} = \begin{pmatrix} {}^t a_1 \\ {}^t a_2 \end{pmatrix}$ must be injective. Otherwise, we may change the basic elements $\{v_{q_1+1}, \dots, v_q, w_{q_1+1}, \dots, w_q\}$ in such a way that, for some $q_1 + 1 \leq r < q$, the last $q - r$ columns of $\tilde{a}$ are null and in that case

$$M = \langle v_{q_1 - r + 1}, \dots, v_q \rangle \oplus \langle x_i, v_j : i = 1, \dots, p; j = 1, \dots, q - r \rangle.$$

Thus $\tilde{a}$ is injective. Change the basis $\{x_i : i = 1, \dots, p\}$ in such a way that

$$a_{12} \cdot v_{q_1+i} = x_i + \frac{\mathrm{i}}{3}(v_{q_1+i} + w_{q_1+i}), \quad i = 1, \dots, q_2.$$

Notice that, as $a_{12}(v_{q_1+i} + w_{q_1+i}) = 0$ for every i and $a_{12}^2 = 0$, then $a_{12} \cdot x_i = 0$, $i = 1, \dots, q_2$. But then

$$M = \bigoplus_{i=1}^{q_2} \langle x_i, v_{q_1+i} \rangle \oplus \langle x_{q_2+1}, \dots, x_p, v_1, \dots, v_{q_1} \rangle.$$

Therefore, if $q_2 > 0$ and M is indecomposable, then $q_1 = 0$, $p = q_2 = 1$, and this gives us the modules in the first item of Lemma 5.9.

Analogously, if $q_1 > 0$, $\tilde{\beta} = \left({}^t\beta_1 \ {}^t\beta_2 \right)$ must be injective, and $q_2 = 0$. If $v_1, \dots, v_p$ are chosen in such a way that $a_{12} \cdot x_i = v_i - w_i$, $i = 1, \dots, p$, then $M = \bigoplus_{i=1}^{p} \langle x_i, v_i \rangle \oplus \bigoplus_{i=p+1}^{q_1} \langle v_i \rangle$ and therefore $p = q_1 = 1$, giving the modules in the third item of the lemma. The modules in the other two items result from these ones by tensoring with S_{sgn}. □

5.3.4. *Tensor product of simple $\mathcal{A}_1$-modules.* Here we compute the tensor product of two given simple $\mathcal{A}_1$-modules, and show that it turns out to be an indecomposable module.

First, we list all of the indecomposable $\mathcal{A}_1$-modules of dimension 4. Notice that if M is such an indecomposable module, then we necessarily must have $M_{|\mathbb{S}_3} = W_\epsilon \oplus W_{\mathrm{sgn}} \oplus W_{\mathrm{st}}$, by Props. 5.7 and 5.10. In the canonical basis, the matrix of a_{12} is given by

$$[a_{12}] = \begin{pmatrix} 0 & \gamma & a & -a \\ \alpha & 0 & b & b \\ \beta & \eta & c & -d \\ -\beta & \eta & d & -c \end{pmatrix},$$

for some $\alpha, \gamma, a, b \in \Bbbk$ and $c = d = \pm\frac{\mathrm{i}}{3}$ or $c = -d = \mathrm{i}$. For every $c = \theta \in \{\pm\mathrm{i}, \pm\frac{\mathrm{i}}{3}\}$ and for each collection $(\alpha, \beta, \gamma, \eta, a, b)$ which defines representation, we denote by $M(\alpha, \beta, \gamma, \eta, a, b)[\theta]$ the corresponding module.

Proposition 5.11.

- *Let $\theta = \pm\frac{\mathrm{i}}{3}$. There are exactly four non-isomorphic indecomposable modules $M(\alpha, \beta, \gamma, \eta, a, b)[\pm\frac{\mathrm{i}}{3}]$. They are defined for $(\alpha, \beta, \gamma, \eta, a, b)$ in the following list:*
  - (i) $(0, 0, 1, 0, 1, 0)$,
  - (ii) $(0, 0, 1, 1, 0, 0)$,
  - (iii) $(1, 0, 0, 0, \mp\frac{2\mathrm{i}}{3}, 1)$,
  - (iv) $(1, 1, 0, \mp\frac{2\mathrm{i}}{3}, 0, 0)$.
- *Let $\theta = \pm\mathrm{i}$. There are exactly four non-isomorphic indecomposable modules $M(\alpha, \beta, \gamma, \eta, a, b)[\pm\mathrm{i}]$. They are defined for $(\alpha, \beta, \gamma, \eta, a, b)$ in the following list:*
  - (i) $(1, 0, 0, 0, 0, 1)$,
  - (ii) $(1, 1, 0, 0, 0, 0)$,
  - (iii) $(0, \mp 2\mathrm{i}, 1, 1, 0, 0)$,
  - (iv) $(0, 0, 1, 0, 1, \mp 2\mathrm{i})$.

The next proof is essentially interpreting the equations (31) in this case.

Proof. We have the following identities

$$\alpha\gamma = \gamma\alpha = 0, \quad \beta a = \beta b = \eta a = \eta b = 0. \tag{33}$$

Assume $c = d = \pm\frac{\mathrm{i}}{3}$, then to the equations listed above we must add:

$$0 = 2\beta c + \alpha\eta = a\alpha + 2cb, \quad 0 = \gamma\beta = b\gamma.$$

We compute the solutions. Notice that $\alpha = 0 \Rightarrow \beta = 0 \Rightarrow b = 0 \Rightarrow \eta a = 0$. Then according to $\eta = 0$ or $a = 0$ we have:

$$\begin{cases} a_{12} \cdot x = 0, \\ a_{12} \cdot y = \gamma x, \\ a_{12} \cdot v = ax + c(v+w) \end{cases} \quad \text{or} \quad \begin{cases} a_{12} \cdot x = 0, \\ a_{12} \cdot y = \gamma x + \eta(v+w), \\ a_{12} \cdot v = c(v+w). \end{cases}$$

Notice that, in any case, we cannot have $\gamma = 0$, otherwise the module would decompose. We may thus assume $\gamma = 1$, changing y by $\frac{1}{\gamma}y$. For the same reason, we cannot have $a = \eta = 0$. In the first case, we may take $a = 1$, changing v by $\frac{1}{a}v$ and in the second case, changing v by ηv we may take $\eta = 1$.

On the other hand, $\gamma = 0 \Rightarrow \alpha \neq 0$; and, according to $\beta = 0$ or $\beta \neq 0$,

$$\beta = 0 \Rightarrow \begin{cases} a_{12} \cdot x = \alpha y, \\ a_{12} \cdot y = 0 \\ a_{12} \cdot v = ax + by + c(v+w), \quad \text{for } a = -2cb\alpha^{-1} \end{cases}$$

$$\beta \neq 0 \Rightarrow \begin{cases} a_{12} \cdot x = \alpha y + \beta(v-w), \\ a_{12} \cdot y = \eta(v+w), \\ a_{12} \cdot v = c(v+w), \qquad \text{for } \eta = -2\beta c\alpha^{-1}. \end{cases}$$

In the first case we may assume $\alpha = b = 1$, and thus $a = -2c$ and, in the second, $\alpha = \beta = 1$, and thus $\eta = -2c$.

Assume now $c = -d = \pm\mathrm{i}$, then to the identities (33) we had we must add:

$$\begin{cases} 0 = 2b\gamma + 2ca = \gamma\beta + 2c\eta \\ 0 = a\alpha = \alpha\eta. \end{cases}$$

We find the solutions:

$$\text{(i)} \begin{cases} a_{12} \cdot x = \alpha y, \\ a_{12} \cdot y = 0, \\ a_{12} \cdot v = by + c(v-w). \end{cases} \qquad \text{(ii)} \begin{cases} a_{12} \cdot x = \alpha y + \beta(v-w), \\ a_{12} \cdot y = 0, \\ a_{12} \cdot v = c(v-w). \end{cases}$$

$$\text{(iii)} \begin{cases} a_{12} \cdot x = \beta(v-w), \\ a_{12} \cdot y = \gamma x + \eta(v+w), \\ a_{12} \cdot v = c(v-w), \\ \beta = -2\eta c\gamma^{-1}. \end{cases} \qquad \text{(iv)} \begin{cases} a_{12} \cdot x = 0, \\ a_{12} \cdot y = \gamma x, \\ a_{12} \cdot v = ax + by + c(v-w), \\ b = -2ca\gamma^{-1}. \end{cases}$$

Therefore, changing conveniently the basis on each case (by scalar multiple of its components), we have the four modules from the second item. $\square$

Let $\operatorname{sgn}: \mathrm{i}\mathbb{R} \to \{\pm 1\}$, $\operatorname{sgn}(\mathrm{i}t) = \operatorname{sgn}(t)$.

Proposition 5.12. *The following isomorphisms hold:*

(i) $S_\epsilon \otimes S \cong S \cong S \otimes S_\epsilon$ *for every simple* $\mathcal{A}_1$*-module* S*;*

(ii) $S_{\mathrm{sgn}} \otimes S_{\mathrm{st}}(\theta) \cong S_{\mathrm{st}}(\vartheta)$*, for* $\theta, \vartheta \in \{\pm\mathrm{i}, \pm\frac{\mathrm{i}}{3}\}$ *with* $\operatorname{sgn}(\theta) = \operatorname{sgn}(\vartheta)$*,* $|\theta| \neq |\vartheta|$*;*

(iii) $S_{\mathrm{st}}(\theta) \otimes S_{\mathrm{sgn}} \cong S_{\mathrm{st}}(\vartheta)$*, for* $\theta, \vartheta \in \{\pm\mathrm{i}, \pm\frac{\mathrm{i}}{3}\}$ *with* $\operatorname{sgn}(\theta) = -\operatorname{sgn}(\vartheta)$*,* $|\theta| \neq |\vartheta|$.

(iv)
- $S_{\mathrm{st}}(\mathrm{i}) \otimes S_{\mathrm{st}}(\mathrm{i}) \cong S_{\mathrm{st}}(-\frac{\mathrm{i}}{3}) \otimes S_{\mathrm{st}}(\frac{\mathrm{i}}{3}) \cong M(0, 2\mathrm{i}, 1, 1, 0, 0)[-\mathrm{i}]$,
- $S_{\mathrm{st}}(\mathrm{i}) \otimes S_{\mathrm{st}}(-\mathrm{i}) \cong S_{\mathrm{st}}(-\frac{\mathrm{i}}{3}) \otimes S_{\mathrm{st}}(-\frac{\mathrm{i}}{3}) \cong M(1, 0, 0, 0, -2\frac{\mathrm{i}}{3}, 1)[\frac{\mathrm{i}}{3}]$,
- $S_{\mathrm{st}}(\mathrm{i}) \otimes S_{\mathrm{st}}(\frac{\mathrm{i}}{3}) \cong S_{\mathrm{st}}(-\frac{\mathrm{i}}{3}) \otimes S_{\mathrm{st}}(\mathrm{i}) \cong M(0, 0, 1, 0, 1, 2\mathrm{i})[-\mathrm{i}]$,
- $S_{\mathrm{st}}(\mathrm{i}) \otimes S_{\mathrm{st}}(-\frac{\mathrm{i}}{3}) \cong S_{\mathrm{st}}(-\frac{\mathrm{i}}{3}) \otimes S_{\mathrm{st}}(-\mathrm{i}) \cong M(1, 1, 0, -2\frac{\mathrm{i}}{3}, 0, 0)[\frac{\mathrm{i}}{3}]$,
- $S_{\mathrm{st}}(-\mathrm{i}) \otimes S_{\mathrm{st}}(\mathrm{i}) \cong S_{\mathrm{st}}(\frac{\mathrm{i}}{3}) \otimes S_{\mathrm{st}}(\frac{\mathrm{i}}{3}) \cong M(1, 0, 0, 0, 2\frac{\mathrm{i}}{3}, 1)[-\frac{\mathrm{i}}{3}]$,
- $S_{\mathrm{st}}(-\mathrm{i}) \otimes S_{\mathrm{st}}(-\mathrm{i}) \cong S_{\mathrm{st}}(\frac{\mathrm{i}}{3}) \otimes S_{\mathrm{st}}(-\frac{\mathrm{i}}{3}) \cong M(0, -2\mathrm{i}, 1, 1, 0, 0)[\mathrm{i}]$,
- $S_{\mathrm{st}}(-\mathrm{i}) \otimes S_{\mathrm{st}}(\frac{\mathrm{i}}{3}) \cong S_{\mathrm{st}}(\frac{\mathrm{i}}{3}) \otimes S_{\mathrm{st}}(\mathrm{i}) \cong M(1, 1, 0, 2\frac{\mathrm{i}}{3}, 0, 0)[-\frac{\mathrm{i}}{3}]$,
- $S_{\mathrm{st}}(-\mathrm{i}) \otimes S_{\mathrm{st}}(-\frac{\mathrm{i}}{3}) \cong S_{\mathrm{st}}(\frac{\mathrm{i}}{3}) \otimes S_{\mathrm{st}}(-\mathrm{i}) \cong M(0, 0, 1, 0, 1, -2\mathrm{i})[\mathrm{i}]$.

Proof. Item (i) is immediate.

We check item (ii): let $\theta \in \{\pm\mathrm{i}, \pm\frac{\mathrm{i}}{3}\}$, $S_{\mathrm{sgn}} = \Bbbk\{z\}$; $S_{\mathrm{st}}(\theta) = \Bbbk\{v, w\}$, $a_{12} \cdot v = cv + dw$. Then $(S_{\mathrm{sgn}} \otimes S_{\mathrm{st}})_{|\mathbb{S}_3} = W_{\mathrm{st}}$ with the canonical basis given by

$$u = z \otimes v - 2z \otimes w, \quad t = 2z \otimes v - z \otimes w,$$

and then

$$a_{12}u = \frac{5c + 4d}{3}u - \frac{4c + 5d}{3}t.$$

Thus, the claim follows according to $c = \pm\mathrm{i}$ or $c = \pm\frac{\mathrm{i}}{3}$.

Item (iii) follows analogously: in this case

$$u = v \otimes z - 2w \otimes z \quad \text{and} \quad a_{12}u = -\frac{5c + 4d}{3}u + \frac{4c + 5d}{3}t.$$

Now, we have to compute $S_{\mathrm{st}}(\theta) \otimes S_{\mathrm{st}}(\vartheta)$, for $\theta, \vartheta \in \{\pm\mathrm{i}, \pm\frac{\mathrm{i}}{3}\}$. Let $S_{\mathrm{st}}(\theta) = \Bbbk\{v, w\}$, $S_{\mathrm{st}}(\vartheta) = \Bbbk\{v', w'\}$, $a = v \otimes v', b = v \otimes w', c = w \otimes v', d = w \otimes w'$. First,

$$W_{\mathrm{st}} \otimes W_{\mathrm{st}} \cong W_\epsilon \oplus W_{\mathrm{sgn}} \oplus W_{\mathrm{st}} = \Bbbk\{x\} \oplus \Bbbk\{y\} \oplus \Bbbk\{v, w\},$$

for $x = 2a - b - c + 2d$, $y = b - c$, $v = a - b - c, w = d - b - c$. Now, if $a_{12} \cdot v = \alpha v + \beta w$ and $a_{12} \cdot v' = \alpha' v' + \beta' w'$, then

$$a_{12} \cdot a = \alpha a + (\beta + \alpha')c + \beta' d, \quad a_{12} \cdot b = \alpha b - \beta' c + (\beta - \alpha')d,$$
$$a_{12} \cdot c = (\alpha' - \beta)a + \beta' b - \alpha c, \quad a_{12} \cdot d = -\beta' a - (\alpha' + \beta)b - \alpha d;$$

and thus

$$\begin{aligned}
a_{12}\cdot x &= (-\alpha-2\beta-2\alpha'-\beta')y+(2\alpha+\beta-\alpha'-2\beta')(v-w),\\
a_{12}\cdot y &= \frac{1}{3}(\alpha+2\beta-2\alpha'-\beta')x+(-2\alpha-\beta+\alpha'+2\beta')(v+w),\\
a_{12}\cdot v &= \frac{1}{6}(2\alpha+\beta+\alpha'+2\beta')x+\frac{1}{2}(-2\alpha-\beta-\alpha'-2\beta')y\\
&\qquad +\frac{1}{3}(\alpha+2\beta-4\alpha'-2\beta')v+\frac{1}{3}(-2\alpha-4\beta+2\alpha'+\beta')w.
\end{aligned}$$

For each $\theta,\vartheta\in\{\pm \mathrm{i}\pm\frac{\mathrm{i}}{3}\}$, we get the identities in item (iv) by inserting the corresponding values of $\alpha,\alpha',\beta,\beta'$. $\square$

Corollary 5.13. *$\mathcal{A}_1$ is not quasitriangular.*

Proof. If H is a quasitriangular Hopf algebra and M,N are H-modules, then $M\otimes N\cong N\otimes M$ as H-modules. We see that this does not hold for $\mathcal{A}_1$, from, for instance, the second item of Prop. 5.12. $\square$

5.3.5. *Projective covers.* Recall that a linear basis for $\mathcal{A}_1$ is given by the set $S=\{xH_t\,|\,x\in X, t\in\mathbb{S}_3\}$, where $X=\{1,a_{12},a_{13},a_{23},a_{12}a_{13},a_{12}a_{23},a_{13}a_{23},a_{13}a_{12},a_{12}a_{13}a_{23},a_{12}a_{13}a_{12},a_{13}a_{12}a_{23},a_{12}a_{13}a_{12}a_{23}\}$ [AG2].

Proposition 5.14. *I_χ is the projective cover of S_χ, $\chi\in\{\epsilon,\mathrm{sgn}\}$.*

Proof. In view of Prop 4.3, we only have to check that I_χ is indecomposable. We work with $\chi=\epsilon$, the other case being analogous, or follows by tensoring with S_{sgn}. Let $e_\epsilon=\sum_{t\in\mathbb{S}_3}H_t\in\mathcal{A}_1$, then it is clear that $\{xe_\epsilon\,|\,x\in X\}$ is a basis of I_ϵ. Moreover, if we change this basis by the following one:

$$\begin{aligned}
&\{e_\epsilon\}\cup\{(a_{12}a_{13}a_{12}a_{23}-a_{12}a_{23})e_\epsilon\}\cup\{(a_{12}+a_{13}+a_{23})e_\epsilon\}\\
&\quad\cup\{(a_{12}a_{13}a_{12}-a_{12}a_{13}a_{23}-a_{13}a_{12}a_{23}-a_{13}-2a_{12})e_\epsilon\}\\
&\quad\cup\{(a_{12}-2a_{13}+a_{23})e_\epsilon,(2a_{23}-a_{12}-a_{13})e_\epsilon\}\\
&\quad\cup\{(a_{13}a_{23}-a_{13}a_{12})e_\epsilon,(a_{12}a_{13}-a_{12}a_{23}+a_{13}a_{23}-a_{13}a_{12})e_\epsilon\}\\
&\quad\cup\{(a_{12}a_{13}+a_{12}a_{23}+a_{13}a_{12})e_\epsilon,(-a_{12}a_{13}+a_{13}a_{23}-a_{13}a_{12})e_\epsilon\}\\
&\quad\cup\{(a_{12}a_{13}a_{12}+2a_{12}a_{13}a_{23}-a_{13}a_{12}a_{23}+a_{12}-a_{13})e_\epsilon,\\
&\qquad(2a_{12}a_{13}a_{12}+a_{12}a_{13}a_{23}+a_{13}a_{12}a_{23}-a_{12}+a_{13})e_\epsilon\}
\end{aligned}$$

then we can see that

$$(I_\epsilon)_{|\mathbb{S}_3}\cong W_\epsilon\oplus W_\epsilon\oplus W_{\mathrm{sgn}}\oplus W_{\mathrm{sgn}}\oplus W_{\mathrm{st}}\oplus W_{\mathrm{st}}\oplus W_{\mathrm{st}}\oplus W_{\mathrm{st}}.$$

Now we deal with the action of a_{12}. Notice that in the first basis, the matrix of a_{12} is $E_{2,1}+E_{5,3}+E_{6,4}+E_{10,7}+E_{9,8}+E_{12,11}$, where $E_{i,j}$ is the matrix whose all its entries are zero except for the (i,j)-th one, which is a 1. It is possible to change the basis in such a way that the decomposition in $\mathbb{S}_3$-simple modules is preserved and the matrix of a_{12} becomes:

$$[a_{12}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2\mathrm{i} & -2\mathrm{i} & 2\mathrm{i} & 2\mathrm{i} & 0 & 0 & 0 & 0 \\ -\frac{1}{12} & 0 & 0 & 0 & \mathrm{i} & \mathrm{i} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{12} & 0 & 0 & 0 & -\mathrm{i} & -\mathrm{i} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{12} & 0 & 0 & 0 & 0 & 0 & -\mathrm{i} & -\mathrm{i} & 0 & 0 & 0 & 0 \\ \frac{1}{12} & 0 & 0 & 0 & 0 & 0 & \mathrm{i} & \mathrm{i} & 0 & 0 & 0 & 0 \\ \frac{1}{12} & 0 & -\frac{\mathrm{i}}{6} & 0 & 0 & 0 & 0 & 0 & \frac{\mathrm{i}}{3} & -\frac{\mathrm{i}}{3} & 0 & 0 \\ -\frac{1}{12} & 0 & -\frac{\mathrm{i}}{6} & 0 & 0 & 0 & 0 & 0 & \frac{\mathrm{i}}{3} & -\frac{\mathrm{i}}{3} & 0 & 0 \\ \frac{1}{12} & 0 & \frac{\mathrm{i}}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\mathrm{i}}{3} & \frac{\mathrm{i}}{3} \\ -\frac{1}{12} & 0 & \frac{\mathrm{i}}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\mathrm{i}}{3} & \frac{\mathrm{i}}{3} \end{bmatrix}.$$

Let $\{x_1, x_2, y_1, y_2, v_1, w_1, v_2, w_2, v_3, w_3, v_4, w_4\}$ be this new basis. Assume $I_\epsilon = U_1 \oplus U_2$, for U_1, U_2 $\mathcal{A}_1$-submodules. Thus, there exists $i = 1, 2$, $\lambda \neq 0, \mu \in \Bbbk$ such that $x = \lambda x_1 + \mu x_2 \in U_i$. Acting with a_{12} we have that $y_1, v_1 + v_2 - v_3 - v_4 \in U_i$. As $y_1 \in U_i$, acting once again with a_{12} we have that also $v_3 - v_4 \in U_i$ and thus $v_3 + v_4 \in U_i$ (again by the action of a_{12}). Therefore $v_3, v_4 \in U_i$ and so $x_2, y_2, x_1, v_1 + v_2 \in U_i$. But then $v_1 - v_2 \in U_i$ and thus $U_i = I_\epsilon$. $\square$

We are left with finding the projective covers $P_{\mathrm{st}}(\theta)$ of the 2-dimensional $\mathcal{A}_1$-modules $S_{\mathrm{st}}(\pm\theta)$, $\theta \in \{\mathrm{i}, \frac{\mathrm{i}}{3}\}$. Since these modules are

$$S_{\mathrm{st}}(\mathrm{i}), \quad S_{\mathrm{st}}(\mathrm{i}) \otimes S_{\mathrm{sgn}}, \quad S_{\mathrm{sgn}} \otimes S_{\mathrm{st}}(\mathrm{i}), \quad \text{and} \quad S_{\mathrm{sgn}} \otimes S_{\mathrm{st}}(\mathrm{i}) \otimes S_{\mathrm{sgn}},$$

see Prop. 5.12, and $P_{\mathrm{st}}(\theta) \cong \mathcal{A}_1 e_{\mathrm{st}}(\theta)$, they will all have the same dimension. Moreover, we will necessarily have $\dim P_{st}(\theta) = 6$, $\forall\, \theta$, by (29).

Proposition 5.15. *Let P be the $\Bbbk\mathbb{S}_3$-module with basis $\{x, y, u, t, v, w\}$, where $\langle x \rangle_{|\mathbb{S}_3} = W_\epsilon$, $\langle y \rangle_{|\mathbb{S}_3} = W_{\mathrm{sgn}}$, $\langle u, t \rangle_{|\mathbb{S}_3} = W_{\mathrm{st}}$, $\langle v, w \rangle_{|\mathbb{S}_3} = W_{\mathrm{st}}$. Then P is an $\mathcal{A}_1$-module via*

$$\Bbbk\{x, y, u, t\} \cong M(0, 2\mathrm{i}, 1, 1, 0, 0)[-\mathrm{i}], \quad a_{12} \cdot v = x - 2\mathrm{i}y + u + t + \mathrm{i}(v - w).$$

Moreover $P = P_{\mathrm{st}}(\mathrm{i})$ is the projective cover of the simple module $S_{\mathrm{st}}(\mathrm{i})$.

As a result, we have $P_{\mathrm{st}}(-\frac{\mathrm{i}}{3}) = P_{\mathrm{st}}(\mathrm{i}) \otimes S_{\mathrm{sgn}}$, $P_{\mathrm{st}}(\frac{\mathrm{i}}{3}) = S_{\mathrm{sgn}} \otimes P_{\mathrm{st}}(\mathrm{i})$ and $P_{\mathrm{st}}(-\mathrm{i}) = S_{\mathrm{sgn}} \otimes P_{\mathrm{st}}(\mathrm{i}) \otimes S_{\mathrm{sgn}}$.

Proof. The matrix of a_{12} in the given basis is

$$[a_{12}] = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -2\mathrm{i} & -2\mathrm{i} \\ 2\mathrm{i} & 1 & -\mathrm{i} & -\mathrm{i} & 1 & -1 \\ -2\mathrm{i} & 1 & \mathrm{i} & \mathrm{i} & 1 & -1 \\ 0 & 0 & 0 & 0 & \mathrm{i} & \mathrm{i} \\ 0 & 0 & 0 & 0 & -\mathrm{i} & -\mathrm{i} \end{pmatrix}.$$

Via the action of H_{13}, H_{23} we define the matrices of a_{13}, a_{23} and then it is easy to check that

$$[H_{12}][a_{12}] = -[a_{12}][H_{12}],$$
$$[a_{12}]^2 = 0$$
$$[a_{12}][a_{13}] + [a_{13}][a_{23}] + [a_{23}][a_{12}] = \mathrm{id}_{6\times 6} - [H_{12}][H_{12}],$$

and thus P is an $\mathcal{A}_1$-module.

Now, it is clear that $U = \Bbbk\{x, y, u, t\}$ is an $\mathcal{A}_1$-submodule and that the canonical projection $\pi : P \twoheadrightarrow P/U$ gives a surjection over $S_{\mathrm{st}}(\mathrm{i})$. Moreover, this surjection is essential. In fact, let $N \subset P$ be an $\mathcal{A}_1$-submodule, such that $N/U \cong S_{\mathrm{st}}(\mathrm{i})$. In particular, there exists $\lambda \neq 0 \in \Bbbk$ such that $\lambda u + v \in P$. Now, $a_{12}(v + \lambda u) = x - 2\mathrm{i}y + (1 - \lambda\mathrm{i})u + (-1 + \lambda\mathrm{i})t + \mathrm{i}(v - w)$, and thus $x, y \in N$. But $x \in N \Rightarrow u, v \in N$ and therefore $N = P$. Consequently, $\pi : P \to P/U$ is essential.

Now, if $(P_{\mathrm{st}}(\mathrm{i}), f)$ is the projective cover of $S_{\mathrm{st}}(\mathrm{i})$, we have the following commutative diagram

$$\begin{tikzcd}
 & & P_{\mathrm{st}}(\mathrm{i}) \arrow[dll, dashed, "g"'] \arrow[d, "f"] \\
P \arrow[r, two heads, "\pi"] & P/U \arrow[r, no head, "\cong"] & S_{\mathrm{st}}(\mathrm{i}).
\end{tikzcd}$$

As π is essential and $\pi(g(P_{\mathrm{st}}(\mathrm{i}))) \cong S_{\mathrm{st}}(\mathrm{i})$ we must have $g(P_{\mathrm{st}}(\mathrm{i})) = P$. But then $\dim P = \dim P_{\mathrm{st}}(\mathrm{i}) = 6$ and thus g is an isomorphism. Therefore, (P, π) is the projective cover of $S_{\mathrm{st}}(\mathrm{i})$. The claim about the projective covers of the other $S_{\mathrm{st}}(\lambda)$'s is now straightforward. □

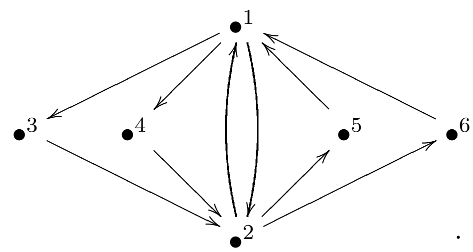
5.3.6. *Representation type of $\mathcal{A}_1$.*

We show that the algebra $\mathcal{A}_1$ is not of finite representation type. From Props. 3.12 and 5.7 it follows that $\mathrm{Ext}^1_{\mathcal{A}_1}(S, S) = 0$ for any simple one-dimensional $\mathcal{A}_1$-module S, and that there is an unique non-trivial extension of S_ϵ by S_{sgn}, namely the $\mathcal{A}_1$-module $M_{\mathrm{sgn},\epsilon}$. The same holds for extensions of S_{sgn} by S_ϵ, considering the $\mathcal{A}_1$-module $M_{\epsilon,\mathrm{sgn}}$. Prop. 3.7 shows that $\mathrm{Ext}^1_{\mathcal{A}_1}(S_{\mathrm{st}}(\lambda), S_{\mathrm{st}}(\mu)) = 0$ for any $\lambda, \mu \in \{\pm\mathrm{i}, \pm\frac{\mathrm{i}}{3}\}$. Now, a non-trivial extension of one of the modules S_ϵ or S_{sgn} by a two dimensional $\mathcal{A}_1$-module $S_{\mathrm{st}}(\lambda)$, or vice versa, must come from a three dimensional indecomposable $\mathcal{A}_1$-module M. We have classified such modules in Lemma 5.9 and we see then that:

$$\dim \mathrm{Ext}^1_{\mathcal{A}_1}(S_\epsilon, S_{\mathrm{st}}(\lambda)) = \dim \mathrm{Ext}^1_{\mathcal{A}_1}(S_{\mathrm{st}}(\lambda), S_{\mathrm{sgn}}) = \begin{cases} 1, & \text{if } \lambda = \pm\mathrm{i}, \\ 0, & \text{if } \lambda = \pm\frac{\mathrm{i}}{3}. \end{cases}$$
$$\dim \mathrm{Ext}^1_{\mathcal{A}_1}(S_{\mathrm{sgn}}, S_{\mathrm{st}}(\lambda)) = \dim \mathrm{Ext}^1_{\mathcal{A}_1}(S_{\mathrm{st}}(\lambda), S_\epsilon) = \begin{cases} 1, & \text{if } \lambda = \pm\frac{\mathrm{i}}{3}, \\ 0, & \text{if } \lambda = \pm\mathrm{i}. \end{cases}$$

Let $\{S_\epsilon, S_{\mathrm{sgn}}, S_{\mathrm{st}}(\mathrm{i}), S_{\mathrm{st}}(-\mathrm{i}), S_{\mathrm{st}}(\frac{\mathrm{i}}{3}), S_{\mathrm{st}}(-\frac{\mathrm{i}}{3})\} = \{1,2,3,4,5,6\}$ be an ordering of the simple $\mathcal{A}_1$-modules. Then the Ext-Quiver of $\mathcal{A}_1$ is:

$Q(\mathcal{A}_1)$:



Proposition 5.16. $\mathcal{A}_1$ *is not of finite representation type.*

Proof. The separation diagram of $\mathcal{A}_1$ is $D_5^{(1)} \coprod D_5^{(1)}$, with $D_5^{(1)}$ the extended affine Dynkin diagram corresponding to the classical Dynkin diagram D_5. By Lemma 4.5 we have that $\mathcal{A}_1/J(\mathcal{A}_1)^2$ (a quotient of $\mathcal{A}_1$) is not of finite representation type (it is, in fact, tame) by Th. 4.4, and so neither is $\mathcal{A}_1$. $\square$

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References

[AG1] Andruskiewitsch, N. and Graña, M., *From racks to pointed Hopf algebras*, Adv. in Math. **178** (2), 177–243 (2003).

[AG2] Andruskiewitsch, N. and Graña, M., *Examples of liftings of Nichols algebras over racks*, Theories d'homologie, representations et algebres de Hopf, AMA Algebra Montp. Announc. 2003, Paper 1, 6 pp. (electronic).

[AHS] Andruskiewitsch, N., Heckenberger, I. and Schneider, H.J., *The Nichols algebra of a semisimple Yetter-Drinfeld module*, arXiv:0803.2430v1.

[ARS] Auslander, M., Reiten, I. and Smalø, S.,*Representation theory of Artin algebras*, Cambridge studies in advanced mathematics **36**.

[AS] N. Andruskiewitsch and H.-J. Schneider, *Pointed Hopf Algebras*, in "New directions in Hopf algebras", 1–68, Math. Sci. Res. Inst. Publ. **43**, Cambridge Univ. Press, Cambridge, 2002.

[AZ] Andruskiewitsch, N. and Zhang, F., *On pointed Hopf algebras associated to some conjugacy classes in* $\mathbb{S}_n$, Proc. Amer. Math. Soc. **135** (2007), 2723–2731.

[CR] Curtis, C. W. and Reiner, I., *Methods of representation theory, with applications to finite groups and orders I*, Wiley Classics Library, (1981).

[GG] García, G. A. and García Iglesias, A., *Pointed Hopf algebras over* $\mathbb{S}_4$. Israel Journal of Math. Accepted. Also available at arXiv:0904.2558v1 [math.QA].

[MS] Milinski, A. and Schneider, H.J., *Pointed indecomposable Hopf algebras over Coxeter groups*, Contemp. Math. 267, 215–236 (2000).

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